AN ENERGY METHOD FOR ROUGH PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We present a well-posedness and stability result for a class of nondegenerate linear parabolic equations driven by rough paths. More precisely, we introduce a notion of weak solution that satisfies an intrinsic formulation of the equation in a suitable Sobolev space of negative order. Weak solutions are then shown to satisfy the corresponding energy estimates which are deduced directly from the equation. Existence is obtained by showing compactness of a suitable sequence of approximate solutions whereas uniqueness relies on a doubling of variables argument and a careful analysis of the passage to the diagonal. Our result is optimal in the sense that the assumptions on the deterministic part of the equation as well as the initial condition are the same as in the classical PDEs theory.

Keywords— rough paths, rough PDEs, energy method, weak solutions
Mathematics Subject Classification — 60H05, 60H15, 35A15, 35D30

1. INTRODUCTION

The so-called variational approach, also known as the energy method, belongs among the most versatile tools in the theory of partial differential equations (PDEs). It is especially useful for nonlinear problems with complicated structure which do not permit the use of (semi-) linear methods such as semigroup arguments, e.g. systems of conservation laws or equations appearing in fluid dynamics. In such cases, solutions are often known or expected to develop singularities in finite time. Therefore, weak (or variational) solutions which can accommodate these singularities provide a suitable framework for studying the behavior of the system in the long run. But even for linear or semi-linear problems, weak solutions are the natural notion of solution in cases where a corresponding mild formulation is not available, for instance due to low regularity of coefficients.

The construction of weak solutions via the energy method relies on basic a priori estimates which can be directly deduced from the equation at hand by considering a suitable test function. The equation is then satisfied in a weak sense, that is, as an equality in certain space of distributions. Within this framework, existence and uniqueness are usually established by separate arguments. The proof of existence often uses compactness of a sequence of approximate solutions. Uniqueness for weak solutions is much more delicate and in some cases even not known. Let us for instance mention problems appearing in fluid dynamics where the questions of uniqueness and regularity of weak solutions remain largely open.

It has been long recognized that addition of stochastic terms to the basic governing equations can be used to model an intrinsic presence of randomness as well as to account for other numerical, empirical or physical uncertainties. Consequently, the field of stochastic partial differential equations massively gained importance over the past decades.

Date: July 24, 2017.
Financial support by the DFG via Research Unit FOR 2402 is gratefully acknowledged.

It relies on the (martingale based) stochastic Itô integration theory, which gave a probabilistic meaning to problems that are analytically ill-posed due to the low regularity of trajectories of the driving stochastic processes. Nevertheless, the drawback appearing already in the context of stochastic differential equations (SDEs) is that the solution map which assigns a trajectory of the solution to a trajectory of the driving signal, known as the Itô map, is measurable but in general lacks continuity. This loss of robustness has obvious negative consequences, for instance when dealing with numerical approximations or in filtering theory.

The theory of rough paths introduced by Lyons [Lyo98] fully overcame the gap between ordinary and stochastic differential equations and allowed for a pathwise analysis of SDEs. The highly nontrivial step is lifting the irregular noise to a bigger space in a robust way such that solutions to SDEs depend continuously on this lifted noise. More precisely, Lyons singled out the appropriate topology on the space of rough paths which renders the corresponding Itô–Lyons solution map continuous as a function of a suitably enhanced driving path. As one of the striking consequences, one can allow initial conditions as well as the coefficients of the equation to be random, even dependent on the entire future of the driving signals - as opposed to the "arrow of time" and the associated need for adaptedness within Itô's theory. In addition, using the rough path theory one can consider drivers beyond the martingale world such as general Gaussian or Markov processes, in contrast to Itô's theory where only semimartingales may be considered.

The rough path theory can be naturally formulated also in infinite dimensions to analyze ODEs in Banach spaces. This generalization is, however, not appropriate for the understanding of rough PDEs. This is due to two basic facts. First, the notion of rough path encodes in a fundamental way the nonlinear effects of time varying signals without any possibility of including signals depending in an irregular way on more parameters. Second, in an infinite dimensional setting the action of a signal (even finite dimensional) is typically described by differential or more generally unbounded operators. Due to these difficulties, attempts at application of the rough path theory in the study rough PDEs have been limited. Namely, it was necessary to avoid unbounded operators by working with mild formulations or Feynman–Kac formulas or transforming the equation in order to absorb the rough dependence into better understood objects such as flow of characteristic curves.

These requirements pose strong limitations on the kind of results one is able to obtain and the proof strategies are very different from classical PDE methods. The most successful approaches to rough PDEs do not even allow to characterize solutions directly but only via a transformation to a more standard PDE problem. However, there has been an enormous research activity in the field of rough path driven PDEs lately and the literature is growing very fast. To name at least a few results relevant for our discussion, we refer the reader to the works by Friz et al. [CF09, CFO11] where flow transformations were applied to fully nonlinear rough PDEs. A mild formulation was at the core of many other works, see for instance Deya–Gubinelli–Tindel [DGT12, GT10] for a semigroup approach to semilinear evolution equations; Gubinelli–Imkeller–Perkowski [GIP15] for the theory of paracontrolled distributions and Hairer [Hai14] for the theory of regularity structures dealing with singular SPDEs.

At this stage, the rough path theory has reached certain level of maturity and it is natural to ask whether one could find rough path analogues to standard PDEs techniques. From this point of view various authors started to develop *intrinsic* formulations of rough PDEs which involve relations between certain distributions associated to the unknown and the driving rough path. Let us mention the work of Gubinelli–Tindel–Torrecilla [GTT14] on viscosity solutions to fully nonlinear rough PDEs, that of Catellier [Cat15] on rough transport equations, Diehl–Friz–Stannat [DFS14] for results based on Feynmann–Kac formula. Finally, Bailleul–Gubinelli [BG15] studied rough transport equations and Deya–Gubinelli–Hofmanová–Tindel [DGHT16a] conservation laws driven by rough paths.

The last two works laid the foundation for the variational approach to rough PDEs: they introduced a priori estimates for rough PDEs based on a new rough Gronwall lemma argument. Consequently, it was possible to derive bounds on various norms of the solution and obtain existence and uniqueness results bypassing the use of the flow transformation or mild formulations. In addition, these techniques were used [DGHT16b] in order to establish uniqueness for reflected rough differential equations, a problem which remained open in the literature as a suitable Gronwall lemma in the context of rough path was missing.

In the present paper, we pursue the line of research initiated in [BG15, DGHT16a]. Our goal is to develop a variational approach to a class of linear parabolic rough PDEs with possibly discontinuous coefficients. To be more precise, we study existence, uniqueness and stability for rough PDEs of the form

$$\begin{cases} \mathrm{d}u - A(t,x)u\,\mathrm{d}t = \big(\sigma^{ki}(x)\partial_i u + \nu^k(x)u\big)\,\mathrm{d}\mathbf{Z}^k \ , \quad \text{on } \mathbb{R}_+\times\mathbb{R}^d\,, \\ u(0)=u_0\,, \end{cases} \tag{1.1}$$

where $\mathbf{Z}\equiv((Z^k)_{0\leq k\leq K},(\mathbb{Z}^{\ell,k})_{1\leq \ell,k\leq K})$ is a geometric rough path of finite $1/\alpha-$variation, with $\alpha\in(1/3,1/2]$. Here and below a summation convention over repeated indexes is used. Regarding the assumptions on the deterministic part of (1.1), we consider an elliptic operator A in divergence form, namely,

$$A(t,x)u=\partial_i\big(a^{ij}(t,x)\partial_j u\big)+b^i(t,x)\partial_i u+c(t,x)u. \tag{1.2}$$

The coefficents $a=(a^{ij})_{1\leq i,j\leq d}$, $b=(b^i)_{1\leq i\leq d}$, c are possibly discontinous. More precisely, we assume that a is symmetric and fulfills a uniform ellipticity condition (see Assumption 2.1). Moreover integrability conditions depending on the dimension d of $\mathbb{R}^d$ are assumed for b,c (see Assumption 2.2). The coefficients in the noise term $\sigma=(\sigma^{ki})_{1\leq k\leq K,1\leq i\leq d}$ and $\nu=(\nu^k)_{1\leq k\leq K}$ possess $W^{3,\infty}$ and $W^{2,\infty}$ regularity, respectively. The initial condition u_0 belongs to L^2.

One can easily see that the above mentioned available approaches to rough PDEs (mild formulation, flow transformation, Feynman–Kac formula) do not apply in this setting. Let us stress that our assumptions on the deterministic part of (1.1) coincide with the classical (deterministic) theory as presented for instance in the book by Ladyzhenskaya, Solonnikov and Ural'tseva [LSU68]. Consequently, there is no doubt that the very natural way to establish existence and uniqueness is the energy method. For completeness, let us mention that problems similar to (1.1) were studied in [CF09, DFS14] (note however that both these references concern equations written in non-divergence form). In comparison to these results, the energy method has clear advantages in several aspects. First, it allows to significantly weaken the required regularity of the coefficients and initial datum. Furthermore, the method does not rely on linearity and thus represents the natural starting point towards more general nonlinear problems.

More precisely, the (unique) solution constructed in [CF09] was obtained as a transformation of a *classical* solution to a certain deterministic equation. For that reason, the coefficients a,b needed to be of class C^2_b with respect to the space variable and the initial condition had the same regularity, whereas the coefficient σ belonged to Lip^γ for some $\gamma > \frac{1}{\alpha} + 3$ (note that $c = 0$, $\nu = 0$ in [CF09]). Besides, the equation was solved in a limiting sense only: a solution is defined as a limit point of classical solutions to the PDE obtained by replacing the driving rough path $\mathbf{Z}$ by its smooth approximation. Uniqueness then corresponded to the fact that there was at most one limit point. We point out that our notion of uniqueness based on an intrinsic formulation of the equation (see Definition 2.2) is stronger as it compares solutions regardless of the way they were constructed.

In the paper [DFS14], an intrinsic weak formulation of an equation of the form (1.1) was introduced and existence of a unique weak solution proved. The approach was based on the Feynman–Kac formula and therefore the equation was solved backward in time. The result required $a, \sigma \in C_b^3$, $b, c \in C_b^1$, $\nu \in C_b^2$ and the terminal condition in C_b^0. Uniqueness was obtained in the class of continuous and bounded weak solutions.

In order to conclude this introductory part, let us be more precise about our approach and results. We recall that, at a heuristic level, the entries of the geometric rough path $\mathbf{Z} \equiv (Z, \mathbb{Z})$ mimic the first and second order iterated integrals

$$\int_s^t \mathrm{d}Z_r \qquad \text{and} \qquad \int_s^t \int_s^r \mathrm{d}Z_{r'} \otimes \mathrm{d}Z_r,$$

respectively. Consequently, it is natural to iterate the equation in order to obtain a generalization of Davie's [Dav07] interpretation of rough differential equations. Namely, we formulate the above equation as

$$\begin{aligned} u_t - u_s = \int_s^t A(r) u_r \,\mathrm{d}r + Z_{st}^k \big(\sigma^{ki}\partial_i + \nu^k\big) u_s \\ + \mathbb{Z}_{st}^{k\ell}\big(\sigma^{ki}\partial_i + \nu^k\big)\big(\sigma^{\ell j}\partial_j + \nu^\ell\big) u_s + o(t-s), \qquad 0 \le s \le t \le T. \end{aligned} \tag{1.3}$$

The equation (1.3) will be solved in a suitable Sobolev space of negative order. Correspondingly, the smallness of the remainder has to be understood in distributional sense as well. Intuitively, a function $u \in C([0,T]; L^2) \cap L^2(0,T; W^{1,2})$ is called a weak solution to (1.1) provided (1.3) holds true as an equality in $W^{-3,2}$. We remark that the corresponding functional setting is similar to the classical theory, i.e. we recognize the usual energy space $C([0,T]; L^2) \cap L^2(0,T; W^{1,2})$ where weak solutions live. Nevertheless, the regularity required from the test functions is higher ($W^{3,2}$ contrary to $W^{1,2}$ in the classical theory). This is a consequence of the low regularity of the driving signal and the consequent need for a higher order expansion.

The first challenge is to derive the corresponding energy estimates leading to the proof of existence. In view of the formulation (1.3), it is clear that the main difficulty is to estimate the remainder term. Indeed, all the other terms in the equation are explicit and can be easily estimated. However, the only information available on the remainder is the equation (1.3) itself. In fact, the definition of a weak solution is to be understood as follows: u is a weak solution to (1.1) provided the 2-index map given by

$$u_{st}^\natural := u_t - u_s - \int_s^t A(r) u_r \,\mathrm{d}r - Z_{st}^k\big(\sigma^{ki}\partial_i + \nu^k\big) u_s - \mathbb{Z}_{st}^{k\ell}\big(\sigma^{ki}\partial_i + \nu^k\big)\big(\sigma^{\ell j}\partial_j + \nu^\ell\big) u_s$$

has finite $(1-\kappa)$-variation, for some $\kappa \in (0,1)$, as a mapping with values in $W^{-3,2}$. It was observed in [BG15, DGHT16a] that there is a trade-off between space and time regularity and a suitable interpolation argument can be used in order to establish sufficient time regularity of the remainder estimated in terms of the energy norm. This is the core of the so-called rough Gronwall lemma argument which in turn yields the desired energy bound for the solution.

We point out that in view of the required regularity of test functions for (1.3), it is remarkable that uniqueness in the class of weak solutions can be established. Indeed, this task requires to test the equation by the weak solution itself and it is immediately seen that the $W^{3,2}$-regularity is far from being satisfied. Nevertheless, as in [BG15, DGHT16a], it is possible perform a tensorization argument which corresponds to the doubling of variables technique known in the context of conservation laws: one considers the equation satisfied by the product $u_t(x)u_t(y)$ and tested by a mollifier sequence $\epsilon^{-d}\psi(\frac{x-y}{\epsilon})$. The core of the proof is then to derive estimates uniform in ϵ in order to be able to pass to the diagonal $x=y$, i.e. to send $\epsilon \to 0$. Once this is done, one obtains the equation for u^2 and proceeds similarly as in the existence part to derive the energy estimate.

Nevertheless, there is a major difference between the derivation of the energy estimates in the existence part and in the proof of uniqueness. Namely, in order to establish a priori estimates needed for existence, one works on the level of sufficiently smooth approximations. This can be done e.g. by mollifying the driving signal and using classical PDE theory. Consequently, deriving the evolution of u^2 is not an issue and can be easily justified. On the other hand, within the proof of uniqueness, the only available regularity is that of weak solutions and the most delicate part is thus to show that u^2 satisfies the right equation.

As discussed above, an important advantage of the rough path theory, as opposed to the stochastic integration theory, is the continuity of the solution map in appropriate topologies. Also in our setting, we obtain the following Wong-Zakai type result which follows immediately from our construction. Let (Z^ϵ) be a sequence of smooth paths whose canonical lifts $\mathbf{Z}^\epsilon \equiv (Z^\epsilon, \mathbb{Z}^\epsilon)$ approximate $\mathbf{Z} \equiv (Z, \mathbb{Z})$ in the rough path sense. Let u^ϵ be the weak solution of (1.1) driven by Z^ϵ obtained by classical arguments. Then we show that u^ϵ converges in $L^2(0,T;L^2_{\mathrm{loc}})$ to u, which is a solution to (1.1) driven by Z.

Outline of the paper. In Section 2, we introduce the main concepts and notations that we use throughout the article, and we state our main results, Theorem 1 and Theorem 2. Section 3 is devoted to the presentation of the main tools necessary to obtain a priori estimates for rough PDEs. The so-called energy inequality, appears at the core of our variational approach. It arises as a consequence of the a priori estimates, Proposition 3.1, applied to the remainder term in the equation governing the evolution of the square of the solution. This is discussed in Section 4. In Section 5 we introduce the above mentioned tensorization argument, which is required in the proof of uniqueness. We present it in a rather general way, motivating the particular choice of function spaces. The uniqueness part, which is treated in Section 6, is the most delicate part of our proof. Finally, the proof of existence as well as stability is presented in Section 7. Several auxiliary results are collected in the Appendix.

2. PRELIMINARIES

2.1. **Notation.** We will denote by $\mathbb{N}_0$ the set of all non-negative integers, that is $\mathbb{N}_0 := \{0,1,2,\dots\}$. Let us recall the definition of the increment operator, denoted by δ. If g is a path defined on $[0,T]$ and $s,t \in [0,T]$ then $\delta g_{st} := g_t - g_s$, if g is a 2-index map defined on $[0,T]^2$ then $\delta g_{s\theta t} := g_{st} - g_{s\theta} - g_{\theta t}$. For a fixed closed time interval $I \subset \mathbb{R}_+$, we denote by Δ, Δ^2 the simplexes

$$\Delta = \Delta_I := \{(s,t) \in I^2 \,,\, s \leq t\} \,, \quad \Delta^2 = \Delta_I^2 := \{(s,\theta,t) \in I^3 \,,\, s \leq u \leq t\} \,. \tag{2.1}$$

We call *control on* I any superadditive map $\omega : \Delta \to \mathbb{R}_+$, that is, for all $(s,\theta,t) \in \Delta^2$ there holds

$$\omega(s,\theta) + \omega(\theta,t) \le \omega(s,t).$$

We say that ω is *regular* provided it vanishes continuously on the diagonal $\{s = t\}$. Given a Banach space E equipped with a norm $|\cdot|_E$, and $a > 0$, we denote by $V_1^{1/a}(I;E)$ the set of paths $g : I \to E$ admitting left and right limits with respect to each of the variables, and such that there exists a regular control $\omega : \Delta \to \mathbb{R}_+$ with

$$|\delta g_{st}|_E \le \omega(s,t)^a\,,$$

for every $(s,t) \in \Delta$. Similarly, we denote by $V_2^{1/a}(I;E)$ the set of 2-index maps $g : \Delta \to E$ such that $g_{tt} = 0$ for every $t \in I$ and

$$|g_{st}|_E \le \omega(s,t)^a\,,$$

for all $(s,t) \in \Delta$, and some regular control ω. Note that $g \in V_1^{1/a}(I;E)$ if and only if $\delta g \in V_2^{1/a}(I;E)$. The corresponding semi-norm in $V_2^{1/a}(I;E)$ is given by

$$|g|_{1/a-\mathrm{var};I;E} := \left(\sup_{\mathfrak{p}\equiv(t_i)\in\mathscr{P}(I)} \sum_{(\mathfrak{p})} |g_{t_i t_{i+1}}|_E^{1/a}\right)^a\,, \tag{2.2}$$

where

$$\mathscr{P}(I) := \Big\{\mathfrak{p} \subset I : \exists l \ge 2, \mathfrak{p} = \{t_1 = \inf I < t_1 < \cdots < t_l = \sup I\}\Big\}$$

is the set of partitions of I, and where, throughout the paper, we use the notational convention:

$$\sum_{(\mathfrak{p})} h_{t_i t_{i+1}} \overset{\text{def}}{=} \sum\nolimits_{i=1}^{\#\mathfrak{p}-1} h_{t_i t_{i+1}} \tag{2.3}$$

for any 2-index element h. By $V_{2,\mathrm{loc}}^{1/a}(I;E)$ we denote the space of maps $g : \Delta \to E$ such that there exists a countable covering $\{I_k\}_k$ of I satisfying $g \in V_2^{1/a}(I_k;E)$ for any k. We also define the set $V_2^{1-}(I;E)$ of negligible remainders as

$$V_2^{1-}(I;E) := \bigcup_{a>1} V_2^{1/a}(I;E),$$

and similarly for $V_{2,\mathrm{loc}}^{1-}(I;E)$.

Furthermore, we denote by $\mathcal{AC}(I;E) \subset V_1^1(I;E)$ the set of *absolutely continuous functions*, that is: $f \in \mathcal{AC}(I;E)$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every non-overlapping family $(s_1,t_1),\dots,(s_n,t_n) \subset I$ with $\sum(t_i - s_i) < \delta$, then

$$\sum_{1\le i\le n} |\delta f_{s_i t_i}|_E < \epsilon\,.$$

Given $\alpha \in (1/3, 1/2]$ and $K \in \mathbb{N}_0$, recall that a continuous (K-dimensional) $1/\alpha$-rough path is a pair $\mathbf{Z} \equiv (Z^k, \mathbb{Z}^{k,\ell})_{1\le k,\ell\le K}$ in $V_2^{1/\alpha}(I;\mathbb{R}^K) \times V_2^{1/(2\alpha)}(I;\mathbb{R}^{K\times K})$ such that Chen's relations hold, namely:

$$\delta Z^k_{s\theta t} = 0\,,\quad \delta \mathbb{Z}^{k\ell}_{s\theta t} = Z^k_{s\theta} Z^\ell_{\theta t}\,,\quad \text{for } (s,\theta,t) \in \Delta^2\,,\quad 1 \le k,\ell \le K, \tag{2.4}$$

and it is called *geometric* if in addition

$$\mathbb{Z}^{k\ell} + \mathbb{Z}^{\ell k} = Z^k Z^\ell\,,\quad 1 \le k,\ell \le K\,. \tag{2.5}$$

We refer the reader to the monographs [FV10, FH14] for a thorough introduction to the rough path theory.

We will consider the usual Lebesgue and Sobolev spaces in the space variable: $L^p \equiv L^p(\mathbb{R}^d)$, $W^{k,p} \equiv W^{k,p}(\mathbb{R}^d)$, for $k \in \mathbb{N}_0$, and $p \in [1, \infty]$, and denote their respective norms by $|\cdot|_{L^p}, |\cdot|_{W^{k,p}}$. The notation $\|\cdot\|_{r,q}$ will be used for the norm in $L^r(I; L^q(\mathbb{R}^d))$, namely:

$$\|f\|_{r,q} := \left(\int_I \left(\int_{\mathbb{R}^d} |f(t,x)|^q \, \mathrm{d}x \right)^{r/q} \mathrm{d}t \right)^{1/r}$$

(note that in contrast to the literature on deterministic PDEs, we write the time variable first, or with a subscript). To emphasize the domain of time integrability we sometimes write $\|\cdot\|_{r,q;I}$. We recall that $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$ is the space of functions f such that for every compact set $K \subset \mathbb{R}^d$ there holds $f|_K \in W^{k,p}(K)$.

We also write $C(I; E)$ for the space of continuous function with values in some Banach space E, endowed with the norm $\|f\|_{C(I;E)} := \sup_{r \in I} |f_r|_E$.

Given Banach spaces X, Y, we will denote by $L(X, Y)$ the space of linear, continuous maps from X to Y, endowed with the operator norm. For f in $X^* := L(X, \mathbb{R})$, we denote the dual pairing by

$${}_{X^*}\langle f, g \rangle_X$$

(i.e. the evaluation of f at $g \in X$). When they are clear from the context, we will simply omit the underlying spaces and write $\langle f, g \rangle$ instead.

2.2. **Unbounded rough drivers.** In the sequel, we call a *scale* any sequence $(\mathcal{G}_k, |\!|\!| \cdot |\!|\!|_k)_{k \in \mathbb{N}_0}$ of Banach spaces such that $\mathcal{G}_{k+1}$ is continuously embedded into $\mathcal{G}_k$, for each $k \in \mathbb{N}_0$.

For each $k \in \mathbb{N}_0$, we will also denote by $\mathcal{G}_{-k}$ the topological dual of $\mathcal{G}_k$, i.e.

$$\mathcal{G}_{-k} := (\mathcal{G}_k)^* . \tag{2.6}$$

Except for the case $\mathcal{G}_k := W^{k,2}$, we do *not* identify $\mathcal{G}_0$ with its dual, hence a (minor) disadvantage of the latter notation is that in general

$$\mathcal{G}_0 \neq \mathcal{G}_{-0} .$$

Definition 2.1. For a given $\alpha \in (1/3, 1/2]$, a pair of 2-index maps $\mathbf{B} \equiv (B, \mathbb{B})$ is called a *continuous unbounded* $1/\alpha$*-rough driver* with respect to the scale $(\mathcal{G}_k)_{k \in \mathbb{N}_0}$, if

(RD1) $B_{st} \in L(\mathcal{G}_{-k}, \mathcal{G}_{-k-1})$ for $k \in \{0,1,2\}$, $\mathbb{B}_{st} \in L(\mathcal{G}_{-k}, \mathcal{G}_{-k-2})$ for $k \in \{0,1\}$, and there exists a regular control $\omega_B : \Delta \to \mathbb{R}_+$ such that

$$\begin{bmatrix} |B_{st}|_{L(\mathcal{G}_{-0}, \mathcal{G}_{-1})} , \quad |B_{st}|_{L(\mathcal{G}_{-2}, \mathcal{G}_{-3})} \leq \omega_B(s,t)^{\alpha} , \\ |\mathbb{B}_{st}|_{L(\mathcal{G}_{-0}, \mathcal{G}_{-2})} , \quad |\mathbb{B}_{st}|_{L(\mathcal{G}_{-1}, \mathcal{G}_{-3})} \leq \omega_B(s,t)^{2\alpha} , \end{bmatrix} \tag{2.7}$$

for every $(s,t) \in \Delta$.

(RD2) Chen's relations hold true, namely, for every $(s, \theta, t) \in \Delta^2$:

$$\delta B_{s\theta t} = 0 , \quad \delta \mathbb{B}_{s\theta t} = B_{\theta t} B_{s\theta} , \tag{2.8}$$

as linear operators on $\mathcal{G}_{-k}$, $k = 0,1,2$, resp. $k = 0,1$.

We will always understand the driver $\mathbf{B}$ *in the sense of distributions*, namely we assume that each $\mathcal{G}_k$ for $k \in \mathbb{N}_0$ is canonically embedded into $\mathscr{D}'(\mathbb{R}^d)$, and that for $u \in \mathcal{G}_{-0}$,

$(s,t)\in\Delta$, the element $B_{st}u$ (resp. $\mathbb{B}_{st}u$) is *defined* as the linear the functional on $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) given by

$$\langle B_{st}u,\phi\rangle=\langle u,B^*_{st}\phi\rangle\,,\quad\forall\phi\in\mathcal{G}_1\,,$$
$$\text{resp.}\quad\langle\mathbb{B}_{st}u,\psi\rangle=\langle u,\mathbb{B}^*_{st}\phi\rangle\,,\quad\forall\phi\in\mathcal{G}_2\,.$$

In the context of (1.1) we let

$$\left[\begin{aligned}&B^*_{st}\phi:=Z^k_{st}\left(-\partial_i(\sigma^{ki}\phi)+\nu^k\phi\right)\,,\\&\mathbb{B}^*_{st}\phi:=\mathbb{Z}^{k\ell}_{st}\Big(\partial_j(\sigma^{\ell j}\partial_i(\sigma^{ki}\phi))-\partial_j(\sigma^{\ell j}\nu^k\phi)-\nu^\ell\partial_i(\sigma^{ki}\phi)+\nu^\ell\nu^k\phi\Big)\end{aligned}\right.\tag{2.9}$$

for a.e. $x\in\mathbb{R}^d$ and every $\phi\in W^{2,\infty}$, assuming that the coefficents σ,ν are regular enough (see the assumption (2.18) below).

2.3. **Assumptions on the coefficients and the main result.** Throughout the paper, we assume that the coefficient $a=(a^{ij})_{1\le i,j\le d}$ corresponding to the highest order terms in A is measurable and such that the following holds.

Assumption 2.1 (Uniform ellipticity condition)**.** The matrix $(a^{ij}(t,x))_{1\le i,j\le d}$ is symmetric, and there exist constants $M,m>0$ such that for a.e. (t,x) :

$$m\sum_{i=1}^d\xi_i^2\le\sum_{1\le i,j\le d}a^{ij}(t,x)\xi_i\xi_j\le M\sum_{i=1}^d\xi_i^2\,,\quad\xi\in\mathbb{R}^d\,.\tag{2.10}$$

We also need assumptions on integrability of the coefficients b and c, depending on the spatial dimension $d\in\mathbb{N}$.

Assumption 2.2. We assume

$$b\in L^{2r}\big(I;L^{2q}(\mathbb{R}^d;\mathbb{R}^d)\big)\quad\text{and}\quad c\in L^r\big(I;L^q(\mathbb{R}^d;\mathbb{R})\big)\,,\tag{2.11}$$

where the numbers $r\in[1,\infty)$ and $q\in(\max(1,\frac{d}{2}),\infty)$ are such that

$$\frac{1}{r}+\frac{d}{2q}\le 1\,.\tag{2.12}$$

The reason for these restrictions will appear in the use of the following interpolation inequality.

Proposition 2.1. *For every f in the space $L^\infty(I;L^2)\cap L^2(I;W^{1,2})$, then f belongs to $L^\rho(I;L^\kappa)$ for every ρ,κ such that*

$$\frac{1}{\rho}+\frac{d}{2\kappa}\ge\frac{d}{4}\quad\textit{and}\quad\left[\begin{array}{lll}\rho\in[2,\infty]\,, & \kappa\in[2,\frac{2d}{d-2}] & \textit{for}\ d>2\\ \rho\in(2,\infty]\,, & \kappa\in[2,\infty) & \textit{for}\ d=2\\ \rho\in[4,\infty]\,, & \kappa\in[2,\infty] & \textit{for}\ d=1\,.\end{array}\right.\tag{2.13}$$

In addition, there exists a constant $\beta>0$ (not depending on f in the above space) such that

$$\|f\|_{L^\rho(I;L^\kappa)}\le\beta\|f\|_{L^\infty(I;L^2)\cap L^2(I;W^{1,2})}\equiv\beta\left(\|\nabla f\|_{L^2(I;L^2)}+\operatorname*{ess\,sup}_{r\in I}|f_r|_{L^2}\right)\,.\tag{2.14}$$

Proof. The proof relies on the complex interpolation (see [Lun09])

$$L^\rho L^\kappa=[L^\infty L^2,L^2L^p]_\theta\,,\tag{2.15}$$

for $\theta := \frac{2}{\rho}$ and $p := 2(1+\rho(\frac{1}{\kappa}-\frac{1}{2}))^{-1}$. Then, thanks to Young Inequality, write

$$\|f\|_{L^\rho L^\kappa} \leq C\|f\|_{L^\infty L^2}^{1-2/\rho}\|f\|_{L^2L^p}^{2/\rho} \leq C'\left(\|f\|_{L^\infty L^2}+\|f\|_{L^2L^p}\right),$$

and (2.14) follows from the Sobolev embedding theorem. For instance when $d>2$, we have $W^{1,2}(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$ if

$$2 \leq p \equiv \frac{2}{1-\rho(\frac{1}{2}-\frac{1}{\kappa})} \leq \frac{2d}{d-2}, \tag{2.16}$$

but from $\frac{1}{\rho}+\frac{d}{2\kappa}\geq\frac{d}{4}$, it holds $\rho\leq\frac{2}{d}(\frac{1}{2}-\frac{1}{\kappa})^{-1}$, and thus $p\leq 2/(1-\frac{2}{d})\equiv\frac{2d}{d-2}$, and since $p\geq 2$, it implies (2.16). The cases $d=1,2$ are left to the reader. For a proof under the stronger assumption that $\frac{1}{\rho}+\frac{d}{2\kappa}=\frac{d}{4}$, we refer to Theorem 2.2 in [LSU68, Chap. II (3.4)]. ■

As an immediate consequence of Proposition 2.1, we have the following. Let r and q be as in (2.12) and let u in $\mathscr{B}$. It is easily seen that (2.12) implies (2.13) for the exponents $\rho := \frac{2r}{r-1}$ and $\kappa := \frac{2q}{q-1}$. Hence, for some universal constant $\beta\equiv\beta(r,q)$, one has

$$\|u\|_{\frac{2r}{r-1},\frac{2q}{q-1}} \leq \beta\|u\|_{\mathscr{B}}. \tag{2.17}$$

Concerning the coefficients of the driver, we assume the following.

Assumption 2.3. The coefficients σ,ν are such that

$$\sigma\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^{d\times K}) \quad \text{and} \quad \nu\in W^{2,\infty}(\mathbb{R}^d,\mathbb{R}^K). \tag{2.18}$$

Throughout the paper, we will extensively make use of the following scales

$$\left[\begin{array}{ll} W^{k,2}(\mathbb{R}^d), & |\!|\!| \cdot |\!|\!|_{k,(2)} := |\cdot|_{W^{k,2}}, \\ W^{k,\infty}(\mathbb{R}^d), & |\!|\!| \cdot |\!|\!|_{k,(\infty)} := |\cdot|_{W^{k,\infty}}, \end{array}\right. \tag{2.19}$$

for $k\in\mathbb{N}_0$, and their corresponding negative-exponent counterparts as in (2.6) (note that usually Sobolev spaces of negative order are defined by the relation $W^{-k,p}=\left(W^{k,p/(p-1)}\right)^*$, except when $p=1,\infty$). Owing to Leibniz rule, it is seen that for a.e. x in $\mathbb{R}^d$ and every (s,t) in Δ :

$$|\nabla^k B^*_{st}\phi| \leq \omega_Z(s,t)^\alpha\left(|\sigma|_{W^{k+1,\infty}}+|\nu|_{W^{k,\infty}}\right)\sum_{0\leq\ell\leq k+1}|\nabla^\ell\phi|, \quad k=0,1,2,$$

whereas

$$|\nabla^k \mathbb{B}^*_{st}\phi| \leq \omega_Z(s,t)^{2\alpha}\left(|\sigma|_{W^{k+2,\infty}}+|\nu|_{W^{k+1,\infty}}\right)\sum_{0\leq\ell\leq k+2}|\nabla^\ell\phi|, \quad k=0,1.$$

The driver $\mathbf{B}=(B,\mathbb{B})$ defined in (2.9) fulfills the properties of Definition 2.1, namely

$$\left[\begin{array}{l} \mathbf{B} \text{ is an } 1/\alpha\text{-unbounded rough driver, with respect to} \\ \text{each of the scales } (W^{k,2})_{k\geq 0} \text{ and } (W^{k,\infty})_{k\geq 0}. \end{array}\right. \tag{2.20}$$

Moreover, we can set

$$\omega_B(s,t) := C\left(|\sigma|_{W^{3,\infty}},|\nu|_{W^{2,\infty}}\right)\omega_Z(s,t) \tag{2.21}$$

for a constant depending on the indicated quantities.

We now need a suitable notion of solution for the problem (1.1). The following definition corresponds to that given in [BG15] (see also [DFS14]).

Definition 2.2. Let $T>0$, $I:=[0,T]$ and $\alpha\in(1/3,1/2]$. Let $\mathbf{B}=(B,\mathbb{B})$ be a continuous $1/\alpha$-unbounded rough driver with respect to a given scale $(\mathcal{G}_k)_{k\in\mathbb{N}_0}$, and let $\mu\equiv\mu_t$ be a path of finite variation in $\mathcal{G}_{-1}$.

A continuous path $g:I\to\mathcal{G}_{-0}$ is called a *weak solution* to the rough PDE

$$\mathrm{d}g=\mathrm{d}\mu+\mathrm{d}\mathbf{B}g \tag{2.22}$$

on $I\times\mathbb{R}^d$, with respect to the scale $(\mathcal{G}_k)_{k\in\mathbb{N}_0}$, if for every $\phi\in\mathcal{G}_3$, and every $(s,t)\in\Delta$, there holds

$$\langle\delta g_{st},\phi\rangle=\langle\delta\mu_{st},\phi\rangle+\langle g_s,B^*_{st}\phi\rangle+\langle g_s,\mathbb{B}^*_{st}\phi\rangle+\langle g^{\natural}_{st},\phi\rangle\,, \tag{2.23}$$

for some $g^\natural\in V^{1-}_{2,\mathrm{loc}}(I;\mathcal{G}_{-3})$.

We have now all in hand to state our main results.

Theorem 1. *Fix $T>0$, $I:=[0,T]$, assume that $u_0\in L^2$, and consider coefficients a,b,c,σ,ν such that Assumptions 2.1-2.2-2.3 hold. There exists a unique weak solution u to* (2.22) *in the sense of Definition 2.2 such that*

$$u\in\mathscr{B}_{0,T}:=C(I;L^2)\cap L^2(I;W^{1,2})\,. \tag{2.24}$$

In addition the following Itô formula holds for the square of u:

$$\langle\delta u^2_{st},\phi\rangle=2\int_s^t\langle Au,u\phi\rangle\,\mathrm{d}r+\langle u_s^2,\hat{B}^*_{st}\phi\rangle+\langle u_s^2,\hat{\mathbb{B}}^*_{st}\phi\rangle+\langle u^{2,\natural}_{st},\phi\rangle\,, \tag{2.25}$$

for every ϕ in $W^{3,\infty}$ and (s,t) in Δ, where $\hat{\mathbf{B}}$ is the unbounded rough driver obtained by replacing ν by $\hat\nu:=2\nu$ in (2.9)*, and where the remainder $u^{2,\natural}$ belongs to $V^{1-}_{2,\mathrm{loc}}(I;(W^{3,\infty})^*)$.*

Finally the $\mathscr{B}$-norm of u is estimated as

$$\|u\|_{\mathscr{B}_{0,T}}\le C\Big(\alpha,T,m,M,\|b\|_{2r,2q},\|c\|_{r,q},\omega_Z,|\sigma|_{W^{3,\infty}},|\nu|_{W^{2,\infty}}\Big)|u_0|_{L^2}\,, \tag{2.26}$$

for a constant depending on the indicated quantities.

Theorem 2. *Under the conditions of Theorem 1, let $\mathcal{P}_{m,M}$ be defined as those coefficents $a^{ij}\in L^\infty(I\times\mathbb{R}^d)$ such that Assumption 2.1 holds, and let $\mathscr{C}^\alpha_g$ be the space of continuous geometric rough paths of finite $1/\alpha$-variation. The solution map*

$$\left[\begin{aligned}&\mathfrak{S}:L^2\times\mathcal{P}_{m,M}\times L^{2r}L^{2q}\times L^rL^q\times W^{3,\infty}\times W^{2,\infty}\times\mathscr{C}^\alpha_g\longrightarrow C\big(I;W^{-1,2}_{\mathrm{loc}}\big)\cap L^2(I;L^2_{\mathrm{loc}})\\&(u_0,a,b,c,\sigma,\nu,\mathbf{Z})\longmapsto\mathfrak{S}(u_0,a,b,c,\sigma,\nu,\mathbf{Z}):=\begin{cases}\textit{the solution given}\\ \textit{by Theorem 1}\end{cases}\end{aligned}\right. \tag{2.27}$$

is continuous.

Remark 2.1. Note that by interpolation it follows from (2.26) and (2.27) that the solution map is continuous in $L^\kappa(I;W^{\gamma,2}_{\mathrm{loc}})$ whenever $\gamma=\theta-(1-\theta)$ and $\kappa\le2/\theta$ for some $\theta\in(0,1)$.

Remark 2.2. The map $u\equiv u_t(x)$ given by Theorem 1 solves $\mathrm{d}u=Au\,\mathrm{d}t+\mathrm{d}\mathbf{B}u$ in the sense that for every ϕ in $W^{3,2}$ and all (s,t) in Δ, it holds

$$\langle\delta u_{st},\phi\rangle=\int_s^t\langle Au,\phi\rangle\,\mathrm{d}r+\int_s^t\langle(\sigma\cdot\nabla+\nu)u,\phi\rangle\,\mathrm{d}\mathbf{Z}\,, \tag{2.28}$$

where the latter makes sense as a rough integral – note that, as a by-product of Proposition 3.1 below, we have that for each $1\le\ell\le K$ the path $t\mapsto\langle(\sigma^\ell\cdot\nabla+\nu^\ell)u_t,\phi\rangle$ *is controlled by* $(Z^k)_{1\le k\le K}$ with Gubinelli derivative $t\mapsto\big\langle(\sigma^\ell\cdot\nabla+\nu^\ell)(\sigma^k\cdot\nabla+\nu^k)u_t,\phi\big\rangle\,,1\le k\le K$.

Remark 2.3 (the case of time-dependent coefficents). It is possible to assume that σ, ν are coefficients depending on space and time, in such a way that the path $t \mapsto \sigma_t \equiv \sigma(t, \cdot)$ is *controlled by* Z in a suitable sense (and similarly for ν), provided one shows counterparts of Propositions 6.1 and 6.2.

Letting $V := W^{3,\infty}(\mathbb{R}^d; \mathbb{R}^{d\times K})$, and assuming for simplicity that $\nu = 0$, consider a V-valued path $\sigma = \sigma_t(x) \in C^\alpha([0,T]; V)$ *controlled by* Z, in the sense that there is some $(\sigma')^k_t(x)$ in $C^\alpha([0,T]; V^K)$ such that

$$\Big((s,t) \in \Delta \mapsto \sigma_s - (\sigma'_s)^k Z^k_{st}\Big) \quad \text{belongs to} \quad V_2^{1/(2\alpha)}([0,T]; V)\,.$$

We can then define the driver $\mathbf{B}$ as the 2-index family of unbounded operators given for φ in $W^{1,2}$ by

$$B_{st}\varphi := \int_s^t \sigma^k \cdot \nabla\varphi \, \mathrm{d}\mathbf{Z}^k = \lim_{\substack{|\mathfrak{p}|\to 0 \\ \mathfrak{p}\in\mathscr{P}([s,t])}} \sum_{(\mathfrak{p})} \sigma^k_{t_i} \cdot \nabla\varphi Z^k_{t_i t_{i+1}} + (\sigma')^{k\ell}_{t_i} \cdot \nabla\varphi \mathbb{Z}^{k\ell}_{t_i t_{i+1}}\,,$$

where we take the limit in the space $W^{-1,2}$, and make use of the summation convention (2.3). The second part of the driver is then defined as the iterated rough integral

$$\mathbb{B}_{st}\varphi := \int_s^t B_{s,r}\, \mathrm{d}B_r(\varphi) = \lim_{\substack{|\mathfrak{p}|\to 0 \\ \mathfrak{p}\in\mathscr{P}([s,t])}} \sum_{(\mathfrak{p})} B_{st_i} B_{t_i t_{i+1}}\varphi + \sigma^k_{t_i} \cdot \nabla\big(\sigma^\ell_{t_i} \cdot \nabla\varphi\big)\mathbb{Z}^{k\ell}_{t_i t_{i+1}}\,,$$

for φ in $W^{2,2}$, where it can be easily checked that the former limit makes sense as an element of $W^{-3,2}$.

With this definition at hand, it is a simple exercise to check that

- $\mathbf{B} \equiv (B, \mathbb{B})$ is an $1/\alpha$-unbounded rough driver on the scale $(W^{k,2})_{k\in\mathbb{N}_0}$;
- any weak solution of the equation "$\mathrm{d}u = Au\,\mathrm{d}t + \mathrm{d}\mathbf{B}u$" (in the sense of Definition 2.2), is is such that that the integral equation (2.28) is fulfilled.

3. Analysis of rough partial differential equations

In this section, we introduce the basic tools necessary for the study of rough PDEs of the form (2.22), namely, the rough Gronwall Lemma and an a priori estimate on the remainder in (2.23). The results were originally introduced in [BG15, DGHT16a] where we also refer the reader for a more detailed introduction. The statements we present below are slightly different than in [BG15, DGHT16a] and hence for readers convenience we also include the proofs. These tools represent the core of our analysis and will be repeatedly used in order to obtain a priori estimates leading to existence as well as uniqueness of weak solutions.

3.1. Rough Gronwall Lemma.

An important ingredient in order to obtain uniform estimates on weak solutions of (1.1) is the following generalized Gronwall-like estimate.

Lemma 3.1 (Rough Gronwall). *Let $G : I \equiv [0,T] \to [0,\infty)$ be a path such that there exist constants $L > 0$, $\kappa \geq 1$, a regular control ω, and a superadditive map φ with:*

$$\delta G_{st} \leq \left(\sup_{s\leq r\leq t} G_r\right)\omega(s,t)^{1/\kappa} + \varphi(s,t)\,, \tag{3.1}$$

for every $(s,t) \in \Delta$ satisfying $\omega(s,t) \leq L$. Then

$$\sup_{0\leq t\leq T} G_t \leq \exp\left(\frac{\omega(0,T)}{\alpha L}\right)\left[G_0 + \sup_{0\leq t\leq T} |\varphi(0,t)| \exp\left(\frac{-\omega(0,t)}{\alpha L}\right)\right] \tag{3.2}$$

where $\alpha := 1 \wedge L^{-1}(2e^2)^{-\kappa}$.

Remark 3.1. A proof under slightly different hypotheses can be found in [DGHT16a]. Note that here we allow for φ which has no sign. This may be relevant in the context of stochasic PDEs, where typically relations such as (3.1) may involve $\varphi(s,t) := M_t - M_s$, the increments of a martingale M.

Proof. Define K to be the largest integer such that

$$\alpha(K-1)L \le \omega(0,T) \le \alpha K L. \tag{3.3}$$

Since the control ω is regular, there exists a sequence $t_0 \equiv 0 < t_1 < \cdots < t_{K-1} < t_K \equiv T$ such that for each k in $\{1,\dots,K-1\}$,

$$\omega(0,t_k) = \alpha L k$$

and, using superadditivity, such that

$$\omega(t_k,t_{k+1}) \le \alpha L\,.$$

Next, for $t \in [0,T]$, we define:

$$G_{\le t} := \sup_{0\le r\le t} G_r\,, \quad H_t := G_{\le t}\exp\left(-\frac{\omega(0,t)}{\alpha L}\right)\,, \quad H_{\le t} := \sup_{0\le r\le t} H_r\,.$$

Fix $t \in [t_{k-1},t_k]$ for some $k \in \{1,\dots,K\}$. Note that since $\alpha \ge 1$, we may apply the estimate (3.1) on each subinterval $[t_i,t_{i+1}]$. Hence using (3.1) and the superadditivity of φ, we write:

$$\begin{aligned}
G_t &= G_0 + \sum\nolimits_{i=0}^{k-2} \delta G_{t_it_{i+1}} + \delta G_{t_{k-1}t}\\
&\le G_0 + (\alpha L)^{1/\kappa}\sum\nolimits_{i=0}^{k-2}\Big(G_{\le t_{i+1}} + \varphi(t_i,t_{i+1})\Big) + (\alpha L)^{1/\kappa} G_{\le t} + \varphi(t_{k-1},t)\\
&\le G_0 + (\alpha L)^{1/\kappa}\sum\nolimits_{i=0}^{k-1} H_{t_{i+1}}\exp\left(\frac{\omega(0,t_{i+1})}{\alpha L}\right) + \varphi(0,t)\\
&\le G_0 + (\alpha L)^{1/\kappa} H_{\le T}\sum\nolimits_{i=0}^{k-1}\exp(i+1) + \varphi(0,t)\\
&\le G_0 + (\alpha L)^{1/\kappa} H_{\le T}\exp(k+1) + \varphi(0,t)\,.
\end{aligned}$$

Whence, using that $\omega(0,t) \ge \omega(0,t_{k-1})$, we have the following estimate of H :

$$\begin{aligned}
H_t &\le \left\{G_0 + |\varphi(0,t)| + (\alpha L)^{1/\kappa}\exp(k+1)H_{\le t}\right\}\exp\left(\frac{-\omega(0,t)}{\alpha L}\right)\\
&\le G_0 + \sup_{t\le T}\left\{|\varphi(0,t)|\exp\left(\frac{-\omega(0,t)}{\alpha L}\right)\right\} + (\alpha L)^{1/\kappa}e^2 H_{\le T}\,,
\end{aligned}$$

According to our definition of α, this yields the bound:

$$H_{\le T} \le \frac{1}{1-e^2(\alpha L)^{1/\kappa}}\left(G_0 + \sup_{t\le T}\left\{|\varphi(0,t)|\exp\left(\frac{-\omega(0,t)}{\alpha L}\right)\right\}\right)\,,$$

from which (3.2) follows. ∎

3.2. **Remainder estimates.** As in the classical theory, the rough Gronwall Lemma presented above is a simple tool that, among others, permits to obtain a priori estimates for rough PDEs of the general form (2.22). It should be stressed however that the most delicate part of this argument is to estimate the remainder in such a way that Lemma 3.1 is indeed applicable. This step is by no means trivial, in particular, due to unboundedness of the involved operators (in the noise terms as well as in the deterministic part of the equation) and the corresponding loss of derivatives. The key observation is that there is a tradeoff between space and time regularity which can be balanced using a suitable interpolation technique. To this end, let us introduce the notion of smoothing operators on a given scale $(\mathcal{G}_k)$.

Definition 3.1. Assume that we are given a scale $(\mathcal{G}_k)_{k\in\mathbb{N}_0}$ with a topological embedding

$$\cup_{k\in\mathbb{N}_0}\mathcal{G}_k \hookrightarrow \mathscr{D}',$$

and let $J_\eta : \mathscr{D}' \to \mathscr{D}', \eta \in (0,1)$, be a family of linear maps. For $m \geq 1$ we say that $(J_\eta)_{\eta\in(0,1)}$ is an *m-step family of smoothing operators on* $(\mathcal{G}_k)$ provided for each $k \in \mathbb{N}_0$:

(J1) J_η maps $\mathcal{G}_k$ onto $\mathcal{G}_{k+m}$, for every $\eta \in (0,1)$,

and there exists a constant $C_J > 0$ such that for any $\ell \in \mathbb{N}_0$ with $|k-\ell| \leq m$:

(J2) if $0 \leq k \leq \ell \leq m+1$, then

$$|J_\eta|_{L(\mathcal{G}_k,\mathcal{G}_\ell)} \leq \frac{C_J}{\eta^{\ell-k}}, \quad \text{for all} \quad \eta \in (0,1)\,; \tag{3.4}$$

(J3) if $0 \leq \ell \leq k \leq m+1$, then

$$|\operatorname{id} - J_\eta|_{L(\mathcal{G}_k,\mathcal{G}_\ell)} \leq C_J \eta^{k-\ell}, \quad \text{for all} \quad \eta \in (0,1)\,. \tag{3.5}$$

Remark 3.2. Whenever the spaces $\mathcal{G}_k$ are Sobolev-like with exponents of integrability different from $1,\infty$, examples of 1-step families of smoothing operators are provided by

$$J_\eta := (\operatorname{id} - \eta^2\Delta)^{-1} \quad \text{or} \quad J_\eta := e^{\eta^2\Delta} \tag{3.6}$$

(under suitable assumptions on the domain of Δ). In $W^{k,2}(\mathbb{R}^d)$ this is easily seen using the Fourier transform: for instance, concerning the first family we can use the inequality

$$\frac{1}{1+(\eta|\xi|)^2} - 1 \leq C_\alpha(\eta|\xi|)^{2\alpha}\,,$$

which holds for every $\alpha \in [0,1]$, and then apply Parseval Identity (the cases $\alpha = \frac{1}{2}, 1$ yield (J3)). Note that smoothing operators similar to the second family above were also extensively used in [OW16].

If $\mathcal{G}_k$ consists of functions ϕ supported on the whole space $\mathbb{R}^d$, one can simply let $J_\eta\phi := \varrho_\eta * \phi$, where ϱ_η is a well-chosen approximation of the identity. The existence of such smoothing families when elements of $\mathcal{G}_k$ are compactly supported is not trivial and is therefore treated in Appendix A.3.

Let us now formulate the main result of this section.

Proposition 3.1 (Estimate of the remainder)**.** *Let $\alpha \in (1/3, 1/2]$ and fix an interval $I \subset [0,T]$. Let $\mathbf{B} = (B, \mathbb{B})$ be a continuous unbounded $1/\alpha$-rough driver on a given scale $\mathcal{G}_k, |\!|\!| \cdot |\!|\!|_k, k \in \mathbb{N}_0$, endowed with a two-step family of smoothing operators (J_η). Consider a continuous drift $\mu \in V_1^1(I;\mathcal{G}_{-1})$ and let ω_μ be a regular control such that*

$$|\!|\!|\delta\mu_{st}|\!|\!|_{-1} \leq \omega_\mu(s,t)\,. \tag{3.7}$$

Let g be a weak solution of (2.22) in the sense of Definition 2.2, such that g is controlled over the whole interval I, that is: $g^\natural \in V_2^{1-}(I;\mathcal{G}_{-3})$.

Then, there exist constants $C, L > 0$, such that if $\omega_B(I) \leq L$, for all $(s,t) \in \Delta_I$,

$$\lvert g^\natural_{st} \rvert_{-3} \leq C \left(\sup_{s\leq r\leq t} \lvert g_r \rvert_{-0} \omega_B(s,t)^{3\alpha} + \omega_\mu(s,t)\omega_B(s,t)^\alpha \right) . \tag{3.8}$$

Furthermore, defining for each $(s,t) \in \Delta$ the first order remainder

$$g^\sharp_{st} := \delta g_{st} - B_{st} g_s \,, \qquad (s,t) \in \Delta_I \,, \tag{3.9}$$

it holds true that

$$\lvert g^\sharp_{st} \rvert_{-1} \leq C \left(\omega_\mu(s,t) + \sup_{s\leq r\leq t} \lvert g \rvert_{-0} \big(\omega_\mu(s,t)^\alpha + \omega_B(s,t)^\alpha \big) \right) , \tag{3.10}$$

$$\lvert g^\sharp_{st} \rvert_{-2} \leq C \left(\omega_\mu(s,t) + \sup_{s\leq r\leq t} \lvert g \rvert_{-0} \omega_B(s,t)^{2\alpha} \right) , \tag{3.11}$$

and finally

$$\lvert \delta g_{st} \rvert_{-1} \leq C \left(\omega_\mu(s,t) + \sup_{s\leq r\leq t} \lvert g \rvert_{-0} \big(\omega_\mu(s,t)^\alpha + \omega_B(s,t)^\alpha \big) \right) , \tag{3.12}$$

for every $(s,t) \in \Delta_I$, such that $(\omega_\mu + \omega_B)(I) \leq L$.

Remark 3.3. It is a classical fact (see [FV10]) that a product

$$\omega_1(s,t)^a \omega_2(s,t)^b$$

where $a + b \geq 1$, and ω_1, ω_2 are controls, is also a control. Consequently, the conclusion (3.8) in the proposition above can be changed to:

$$\omega_\natural(s,t) \leq C \left(\sup_{s\leq r\leq t} \lvert g_r \rvert_{-0} \omega_B(s,t)^{3\alpha} + \omega_\mu(s,t)\omega_B(s,t)^\alpha \right) , \tag{3.13}$$

where for a given $(s,t) \in \Delta$ we define

$$\omega_\natural(s,t) := \inf\{\omega(s,t) : \omega \in \mathfrak{C}_{s,t}\} \tag{3.14}$$

$$\mathfrak{C}_{s,t} := \left\{ \omega : \Delta_{[s,t]} \to \mathbb{R}_+ \,, \text{control} \;\middle|\; \forall (\theta,\tau) \in \Delta_{[s,t]},\; \omega(\theta,\tau) \geq \lvert g^\natural_{\theta\tau} \rvert_{-3} \right\} . \tag{3.15}$$

Indeed, this is justified by the following basic observation.
Claim. The map $\omega_\natural : \Delta \to \mathbb{R}_+$ is a control.

Proof of claim. For $(s,\theta,t) \in \Delta^2$, since both $\mathfrak{C}_{s,\theta}, \mathfrak{C}_{\theta,t}$ contain $\mathfrak{C}_{s,t}$, we have by definition:

$$\omega_\natural(s,\theta) + \omega_\natural(\theta,t) \leq \omega(s,\theta) + \omega(\theta,t) \leq \omega(s,t) \,, \tag{3.16}$$

for every $\omega \in \mathfrak{C}_{s,t}$.

Taking the infimum in (3.16), the claim follows. ∎

Now, since the r.h.s. of (3.8) is a control, according to the above claim, then (3.13) clearly holds.

We now have all in hand to prove Proposition 3.1.

Proof of Proposition 3.1. Proof of (3.8). To estimate the remainder $g^{\natural}_{st}$, we apply δ to (2.23) and use Chen's relations (2.8), leading to

$$\begin{aligned}\delta g^{\natural}_{s\theta t} &= B_{\theta t}\delta g_{s\theta} - B_{\theta t}B_{s\theta}g_s + \mathbb{B}_{\theta t}\delta g_{s\theta}\\ &= B_{\theta t}g^{\sharp}_{s\theta} + \mathbb{B}_{\theta t}\delta g_{s\theta}\\ &=: \mathcal{T}_{\sharp} + \mathcal{T}_{\delta}\,,\end{aligned}\tag{3.17}$$

for every $(s,\theta,t)\in\Delta^2$. Note that by definition of $g^{\sharp}$ in (3.9) and the original equation (2.23), it holds

$$g^{\sharp}_{s\theta} \equiv \delta g_{s\theta} - B_{s\theta}g_s = \delta\mu_{s\theta} + \mathbb{B}_{s\theta}g_s + g^{\natural}_{s\theta}\tag{3.18}$$

hence it is both an element of $\mathcal{G}_{-1}$ and $\mathcal{G}_{-2}$, (with corresponding different time regularities). This basic fact will be exploited in the sequel, in order to apply Proposition A.1.

In (3.17), test against $\phi\in\mathcal{G}_3$ such that $|\phi|_3\leq 1$. Substituting (3.18) into (3.17) and then making use of (J_η), there comes

$$\langle\mathcal{T}_{\sharp},\phi\rangle \equiv \langle\delta\mu_{s\theta} + \mathbb{B}_{s\theta}g_s + g^{\natural}_{s\theta}, B^*_{\theta t}J_\eta\phi\rangle + \langle\delta g_{s\theta} - B_{s\theta}g_s, B^*_{\theta t}(\mathrm{id}-J_\eta)\phi\rangle\,.$$

Each term above can be estimated using the bounds on $\mathbf{B}$ as well as ω_μ and the estimates (3.4). Denoting for simplicity

$$G := \sup_{s\leq r\leq t}|g_r|_{-0}\,,\tag{3.19}$$

we have:

$$\begin{aligned}\langle\mathcal{T}_{\sharp},\phi\rangle &\leq \omega_\mu|B^*_{s\theta}J_\eta\phi|_1 + \langle g_s, \mathbb{B}^*_{s\theta}B^*_{\theta t}J_\eta\phi\rangle + \langle g^{\natural}_{s\theta}, B^*_{\theta t}J_\eta\phi\rangle\\ &\qquad + \langle\delta g_{s\theta}, B^*_{\theta t}(\mathrm{id}-J_\eta)\phi\rangle + \langle g_s, B^*_{s\theta}B^*_{\theta t}(\mathrm{id}-J_\eta)\phi\rangle\\ &\leq C_J\Big(\omega_\mu\omega_B^\alpha + G\omega_B^{3\alpha} + \frac{\omega_\natural\omega_B^\alpha}{\eta} + 2G\omega_B^\alpha\eta^2 + G\omega_B^{2\alpha}\eta\Big)\,.\end{aligned}\tag{3.20}$$

We now choose η that equilibrates the various terms, namely

$$\eta := 4C_J|\Lambda|\omega_B(s,t)^\alpha\,,\tag{3.21}$$

where $|\Lambda|$ is the constant from the Sewing Lemma, see Proposition A.1. Provided $(s,t)\in\Delta_I$ are sufficiently close to each other, e.g. assuming

$$\omega_B(s,t) < L := \left(\frac{1}{4C_J|\Lambda|}\right)^{1/\alpha}\tag{3.22}$$

then η belongs to $(0,1)$. We end up with the inequality

$$|\mathcal{T}_{\sharp}|_{-3} \leq C\Big(\omega_\mu\omega_B^\alpha + G\omega_B^{3\alpha}\Big) + \frac{\omega_\natural}{4|\Lambda|}\tag{3.23}$$

for some constant $C>0$ depending only on $|\Lambda|$ and C_J. The previous computations also show that for $\phi\in\mathcal{G}_1$ with $|\phi|_1\leq 1$:

$$\begin{aligned}\langle g^{\sharp},\phi\rangle &\leq \omega_\mu|J_\eta\phi|_1 + G\omega_B^{2\alpha}|J_\eta\phi|_2 + \omega_\natural|J_\eta\phi|_3\\ &\qquad + |\delta g|_{-0}|(\mathrm{id}-J_\eta)\phi|_0 + G|B^*(\mathrm{id}-J_\eta)\phi|_0\\ &\leq C_J\Big(\omega_\mu + G\frac{\omega_B^{2\alpha}}{\eta} + \frac{\omega_\natural}{\eta^2} + 2G\eta + G\omega_B^\alpha\Big)\end{aligned}\tag{3.24}$$

where we have used again (3.4). Choosing η as in (3.21), we obtain that $g^\sharp \in V_2^\alpha(I;\mathcal{G}_{-1})$, together with the bound:

$$\|g^\sharp_{st}\|_{-1} \leq C\big(\omega_\mu + G\omega_B^\alpha\big) + \frac{\omega_\natural}{4|\Lambda|\omega_B^{2\alpha}}\,. \tag{3.25}$$

Now, for the second term in (3.17) we can use (3.25): taking $\phi \in \mathcal{G}_3$ with $\|\phi\|_3 \leq 1$, there comes

$$\begin{aligned}\langle \mathcal{T}_\delta, \phi\rangle &\equiv \langle g^\sharp + Bg_s, \mathbb{B}^*\phi\rangle \\ &\leq \|g^\sharp_{st}\|_{-1}\|\mathbb{B}^*\phi\|_1 + \|g_s\|_{-0}\|B^*\mathbb{B}^*\phi\|_0 \\ &\leq C\big(\omega_\mu\omega_B^{2\alpha} + G\omega_B^{3\alpha}\big) + \frac{\omega_\natural}{4|\Lambda|} + G\omega_B^{3\alpha}\,.\end{aligned} \tag{3.26}$$

From the bounds (3.26) and (3.23), we obtain

$$\|\delta g^\natural_{s\theta t}\|_{-3} \leq C\left(\omega_\mu(s,t)\omega_B(s,t)^\alpha + G\omega_B(s,t)^{3\alpha}\right) + \frac{\omega_\natural(s,t)}{2|\Lambda|}\,,$$

for some absolute constant $C > 0$. We are now in position to apply the Sewing Lemma, Proposition A.1, so that $g^\natural = \Lambda\delta g^\natural$ and moreover for all $(s,t) \in \Delta_I$:

$$\|g^\natural_{st}\|_{-3} \leq \omega'_\natural \equiv C\left(\omega_\mu(s,t)\omega_B(s,t)^\alpha + G\omega_B(s,t)^{3\alpha}\right) + \frac{1}{2}\omega_\natural(s,t)\,.$$

Since $\omega_\natural$ is taken to be the smallest control $\omega'_\natural$ such that the inequality above holds (see Remark 3.3), we eventually obtain

$$\|g^\natural_{st}\|_{-3} \leq 2C\left(\omega_\mu(s,t)\omega_B(s,t)^\alpha + G\omega_B^{3\alpha}(s,t)\right)\,,$$

which proves (3.8).

Proof of (3.10). From (3.24) and (3.8), there holds (again we omit the time indexes):

$$\langle g^\sharp, \phi\rangle \leq C\left(\omega_\mu + G\left(\frac{\omega_B^{2\alpha}}{\eta} + \omega_B^\alpha + \eta\right) + \frac{1}{\eta^2}\left(\omega_\mu\omega_B^\alpha + G\omega_B^{3\alpha}\right)\right)\|\phi\|_1\,,$$

whence provided $(\omega_\mu + \omega_B)(I) < L$ is small enough, taking $\eta := (\omega_\mu + \omega_B)^\alpha \in (0,1)$, we end up with the a priori estimate

$$\|g^\sharp_{st}\|_{-1} \leq C\left(\omega_\mu + G\big(\omega_\mu^\alpha + \omega_B^\alpha\big)\right)\,,$$

for $(s,t) \in \Delta_I$ (here we have used the trivial bounds $\omega_B \leq \omega_\mu + \omega_B$, $1-\alpha > \alpha$, as well as $(\omega_\mu + \omega_B)^\alpha \leq C_\alpha(\omega_\mu^\alpha + \omega_B^\alpha)$).

Proof of (3.12) Writing that $\delta g = g^\sharp + Bg$, we see that the same bound holds for δg instead of $g^\sharp$, namely

$$\|\delta g_{st}\|_{-1} \leq C\left(\omega_\mu + G\big(\omega_\mu^\alpha + \omega_B^\alpha\big)\right)$$

(with another such universal constant C).

Proof of (3.11) Proceeding similarly, we have

$$\langle g^\sharp, \phi\rangle \leq C\Big(\omega_\mu + G\left(\omega_B^{2\alpha} + \eta^2 + \omega_B^\alpha\eta\right) + \frac{1}{\eta}\left(\omega_\mu\omega_B^\alpha + G\omega_B^{3\alpha}\right)\Big)\|\phi\|_2\,,$$

whence taking $\eta := \omega_B^\alpha$, we end up with

$$\|g^\sharp_{st}\|_{-2} \leq C\left(\omega_\mu + G\omega_B^{2\alpha}\right)$$

for some universal $C > 0$. ■

Remark 3.4 (On the link between weak solutions and the notion of controlled path). Following Gubinelli's approach on rough paths [Gub04], it would be natural in this setting to define the set $\mathscr{D}_B$ of *controlled paths* as those couples g, g' in $V_1^{1/\alpha}(I;\mathcal{G}_{-1})$, such that the first order remainder

$$(s,t) \in \Delta \mapsto g^\sharp_{st} := \delta g_{st} - B_{st} g'_s \tag{3.27}$$

defines an element of $V_{2,\mathrm{loc}}^{1/(2\alpha)}(I;\mathcal{G}_{-2})$ (meaning that a cancellation occurs in (3.27)).

If g denotes a weak solution of (2.22), in the sense of Definition 2.2, we have in fact $(g, g') \in \mathscr{D}_B$ with $g' = g$. Therefore, given $(\mathcal{G}_k), (J_\eta), \mu$, and $\mathbf{B}$ as in Proposition 3.1, we can alternatively define a weak solution to (2.22) as an element (g, g) of $\mathscr{D}_B$ such that (2.23) holds, i.e. a continuous path $g : [0, T] \to \mathcal{G}_{-0}$ such that

$$\begin{cases} \delta g \in V_2^{1/\alpha}(I, \mathcal{G}_{-1}) \,, \\ g^\sharp \equiv \delta g - Bg \in V_{2,\mathrm{loc}}^{1/(2\alpha)}(I, \mathcal{G}_{-2}) \,, \\ g^\natural \equiv \delta g - Bg - \mathbb{B}g - \delta\mu \in V_{2,\mathrm{loc}}^{1/(3\alpha)}(I, \mathcal{G}_{-3}) \,. \end{cases}$$

4. The energy inequality

In this section we assume that the driving path z is *smooth* and we establish an estimate on the $\mathscr{B}$-norm of a weak solution to (2.22) which only depends on the rough path norm of the corresponding canonical lift $\mathbf{Z}$ of z. However it should be noted that the conclusion of Proposition 4.1 below remains true provided the square u^2 satisfies the equation (2.25), which will be shown to hold for any weak solution u, see Section 6.

4.1. The main statement.

Using the standard theory for non-degenerate parabolic PDEs (see [LSU68, Chap. III]), we know that there exists a unique u in the Banach space $\mathscr{B}$ (note this space is denoted by $V_2^{1,0}$ in the latter reference), solving the the evolution problem

$$\frac{\partial u}{\partial t} - Au = \left(\sigma^{ki}\partial_i u + \nu^k u\right)\dot{z}^k \,, \quad u_0 \in L^2 \,, \tag{4.1}$$

in the sense that

$$-\iint_{I\times\mathbb{R}^d} u\partial_t\eta \,\mathrm{d}t\,\mathrm{d}x + \iint_{I\times\mathbb{R}^d} \left(a^{ij}\partial_j u\partial_i\eta - b^i\partial_i u\eta - cu\eta\right)\mathrm{d}t\,\mathrm{d}x = \iint_{I\times\mathbb{R}^d} \left(\sigma^{ki}\partial_i\eta + \nu^k u\eta\right)\dot{z}^k \,\mathrm{d}t\,\mathrm{d}x \,, \tag{4.2}$$

for every test function η in the Sobolev space

$$\mathscr{W}_2^{1,1}(I\times\mathbb{R}^d) := \{\eta \in L^2(I\times\mathbb{R}^d) : \nabla\eta, \partial_t\eta \in L^2(I\times\mathbb{R}^d)\} \,,$$

and such that η vanishes, in the sense of traces at $t = T$ and $t = 0$.

Our aim is to prove following.

Proposition 4.1 (Energy inequality)**.** *Consider a* smooth path z, *together with its canonical geometric lift* $\mathbf{Z} \equiv (Z, \mathbb{Z})$, *and let* ω_Z *be the smallest control such that for each* $(s,t) \in \Delta$

$$\omega_Z(s,t) \ge |Z_{st}|^{1/\alpha} + |\mathbb{Z}_{st}|^{1/(2\alpha)} \,.$$

Then every weak solution of (1.1) *satisfies*

$$\sup_{0\le t\le T} |u_t|^2_{L^2} + \int_0^T |\nabla u_r|^2_{L^2}\,\mathrm{d}r \le C|u_0|^2 \,, \tag{4.3}$$

for a constant $C>0$ depending on the quantities ω_Z, $|\sigma|_{W^{3,\infty}}$, $|\nu|_{W^{2,\infty}}, m, M$, and $\|b\|_{2r,2q}$, $\|c\|_{r,q}$, but not on the individual element u in $\mathscr{B}$.

Although u does not belong to $\mathscr{W}_2^{1,1}$ a priori, by considering time averages of the form

$$u_h(t,x) := \frac{1}{h}\int_t^{t+h} u(t,\tau)\,\mathrm{d}\tau\,,$$

(extended by zero if $t\notin[0,T-h]$) and passing to the limit $h\to 0$, it is seen that in (4.2) we can formally test against

$$\eta(r,x) := \mathbf{1}_{[s,t]}(r)\phi(x)u(r,x)$$

with $\phi\in W^{1,\infty}$, (see the equality (2.13) in [LSU68, Chap. III, S2] for the case where $\eta := \mathbf{1}_{[s,t]}u$, the proof being identical for η as above). This yields, for each (s,t) in Δ, and every ϕ in $W^{1,\infty}$:

$$\int_{\mathbb{R}^d}((u_t)^2-(u_s)^2)\phi\,\mathrm{d}x = 2\iint_{[s,t]\times\mathbb{R}^d}\big(-a^{ij}\partial_j u\partial_i(u\phi)+b^i\partial_i uu\phi+cu^2\phi\big)\,\mathrm{d}r\,\mathrm{d}x \\ + \iint_{[s,t]\times\mathbb{R}^d}\big(\sigma^{ki}\partial_i(u^2)\phi+2\nu^k u^2\phi\big)\,\dot z^k\,\mathrm{d}r\,\mathrm{d}x\,. \tag{4.4}$$

4.2. **Proof of Proposition 4.1.** We are going to make use of the tools presented in Section 3. More precisely, we will show that

- suitable estimates relative to the scale $(W^{k,\infty})_{k\in\mathbb{N}_0}$ hold for the drift part of (4.4), i.e. for
$$\int_0^{\cdot} uAu\,\mathrm{d}r$$
understood as a linear functional on $W^{1,\infty}$;
- equation (4.4) implies that $\mathrm{d}(u^2) = 2\,\mathrm{d}\big(\int uAu\,\mathrm{d}r\big)+\mathrm{d}\hat{\mathbf{B}}(u^2)$ holds in the sense of Definition 2.2.

An important observation is the following Lemma. For convenience, and because it will be useful in the proof of Theorem 2, we also include bounds on the drift term of u in (4.2).

Lemma 4.1. *Given u in $\mathscr{B}$, define the drift terms*

$$\langle\lambda_t,\phi\rangle := \int_0^t\langle A_r u_r,\phi\rangle\,\mathrm{d}r \equiv \iint_{[0,t]\times\mathbb{R}^d}(-a_r^{ij}\partial_i u_r\partial_j\phi+b_r^i\partial_i u_r\phi+c_r u_r\phi)\,\mathrm{d}r\,\mathrm{d}x\,, \tag{4.5}$$

for ϕ in $W^{1,2}$, $(s,t)\in\Delta$, and

$$\langle\mu_t,\phi\rangle := 2\int_0^t\langle u_rA_ru_r,\phi\rangle\,\mathrm{d}r \equiv 2\iint_{[0,t]\times\mathbb{R}^d}\Big(-a_r^{ij}\partial_i u_r\partial_j u_r\phi \\ - u_ra_r^{ij}\partial_i u_r\partial_j\phi+b_r^i\partial_i u_ru_r\phi+c_r(u_r)^2\phi\Big)\,\mathrm{d}r\,\mathrm{d}x\,, \tag{4.6}$$

for ϕ in $W^{1,\infty}$. Then, there exist regular controls $\omega_\mu,\omega_\lambda$, and a constant $C>0$, the latter three being dependent on

$$T,r,q,M,\|b\|_{2r,2q},\|c\|_{r,q}\,, \tag{4.7}$$

but not on u in the space $\mathscr{B}$, such that

$$\left|\!\left|\!\left|\delta\lambda_{st}\right|\!\right|\!\right|_{-1,(2)} \leq \omega_\lambda(s,t) \leq C\left(1+\|u\|^2_{\mathscr{B}_{s,t}}\right), \tag{4.8}$$

$$\left|\!\left|\!\left|\delta\mu_{st}\right|\!\right|\!\right|_{-1,(\infty)} \leq \omega_\mu(s,t) \leq C\|u\|^2_{\mathscr{B}_{s,t}}, \tag{4.9}$$

for every (s,t) in Δ.

Remark 4.1. Taking a, b, c such that Assumptions 2.1-2.2 hold true, and u in $\mathscr{B}$, the following quantities are controls

$$\begin{cases} \mathbf{a}(s,t) := M\left(\|\nabla u\|^2_{2,2;[s,t]} + \|u\nabla u\|_{1,1;[s,t]}\right) \\ \mathbf{b}(s,t) := \left(\|b\|_{2r,2q;[s,t]}\right)^{2r}, \\ \mathbf{c}(s,t) := \left(\|c\|_{r,q;[s,t]}\right)^{r}, \\ \mathbf{u}(s,t) := \left(\|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]}\right)^{\frac{2r}{r-1}}. \end{cases} \tag{4.10}$$

This will be extensively used in the sequel.

Proof of Lemma 4.1. Proof of (4.8). Take any $\phi \in W^{1,2}$. For u in $\mathscr{B}$, we have

$$-\iint_{[s,t]\times\mathbb{R}^d} a^{ij}\partial_j u \partial_i \phi \,\mathrm{d}r\,\mathrm{d}x \leq M\|\nabla u\|_{1,2;[s,t]} \left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{1,(2)} \leq M(t-s)^{1/2}\|\nabla u\|_{2,2;[s,t]}\left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{1,(2)}.$$

By the equality

$$\frac{1}{2r}+\frac{1}{2}+\frac{r-1}{2r} = 1, \tag{4.11}$$

(and similarly for q), Hölder's inequality yields:

$$\iint_{[s,t]\times\mathbb{R}^d} b^i\partial_i u\phi\,\mathrm{d}r\,\mathrm{d}x \leq \|b\|_{2r,2q;[s,t]}\|\nabla u\|_{2,2;[s,t]}(t-s)^{\frac{r-1}{2r}}|\phi|_{L^{\frac{2q}{q-1}}}. \tag{4.12}$$

Now, in dimension one and two, $W^{1,2}$ embeds into every L^p space for $p\in[1,\infty)$, so the term $|\phi|_{L^{\frac{2q}{q-1}}}$ is bounded by a constant times $\left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{1,(2)}$. For $d>2$, since by assumption

$$q > \max(1,\frac{d}{2}) = \frac{d}{2},$$

it is seen that

$$\frac{2q}{q-1} < \frac{2d}{d-2} =: p^*.$$

By the the Sobolev embedding theorem, we have

$$W^{1,2} \hookrightarrow L^{p^*} \subset L^{\frac{2q}{q-1}}.$$

Hence, in both cases, we see from (4.12) that

$$\iint_{[s,t]\times\mathbb{R}^d} b^i\partial_i u\phi \leq \|b\|_{2r,2q;[s,t]}\|\nabla u\|_{2,2;[s,t]}(t-s)^{\frac{r-1}{2r}}\left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{1,(2)}.$$

Similarly, we have for the last term

$$\begin{aligned}\iint_{[s,t]\times\mathbb{R}^d} cu\phi\,\mathrm{d}r\,\mathrm{d}x \leq & \|c\|_{r,q;[s,t]}\|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]}|\phi|_{L^{\frac{2q}{q-1}}}(t-s)^{\frac{r-1}{2r}} \\ \leq & \|c\|_{r,q;[s,t]}\|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]}(t-s)^{\frac{r-1}{2r}}\left|\!\left|\!\left|\phi\right|\!\right|\!\right|_{1,(2)}.\end{aligned}$$

This yields the inequality

$$\begin{aligned}\| \delta\lambda_{st} \|_{-1,(2)} \leq (t-s)^{1/2}\mathbf{u}(s,t)^{1/2} + \mathbf{b}(s,t)^{1/(2r)}\mathbf{a}(s,t)^{1/2}(t-s)^{\frac{r-1}{2r}} \\ + \mathbf{c}(s,t)^{1/r}\mathbf{u}(s,t)^{\frac{r-1}{2r}}(t-s)^{\frac{r-1}{2r}} .\end{aligned} \tag{4.13}$$

By (4.11) together with Remark 3.3, we see that the r.h.s. above is in fact a control, which proves the first inequality in (4.8).

Now, from $\|u\nabla u\|_{1,1} \leq (t-s)^{1/2}\|u\|_{\infty,2}\|\nabla u\|_{2,2}$ it is clear that

$$\mathbf{a}(s,t)^{1/2} \leq C(M,T)\|u\|_{\mathscr{B}_{s,t}} , \tag{4.14}$$

whereas for the other terms, we use (2.17), so that $\mathbf{u}(s,t)^{\frac{r-1}{2r}} \equiv \|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]} \leq \beta\|u\|_{\mathscr{B}_{s,t}}$. The conclusion follows: using (4.13)-(4.14) we have

$$\| \delta\lambda_{st} \|_{-1,(2)} \leq C(M,\|b\|_{2r,2q},\|c\|_{r,q},T,r,q)\left(1+\|u\|^2_{\mathscr{B}_{s,t}}\right) ,$$

which proves the second part.

Proof of (4.9). Take any ϕ in $W^{1,\infty}$. From Hölder Inequality, it holds true that

$$\iint_{[s,t]\times\mathbb{R}^d} -a^{ij}\partial_j u\partial_i(u\phi) \leq M\left(\|\nabla u\|^2_{2,2;[s,t]} + \|u\nabla u\|_{1,1;[s,t]}\right) \|\phi\|_{1,(\infty)} . \tag{4.15}$$

Now, because of (4.11) we have

$$\iint_{[s,t]\times\mathbb{R}^d} |u||b^i||\partial_i u||\phi| \,\mathrm{d}r\,\mathrm{d}x \leq \|b\|_{2r,2q;[s,t]}\|\nabla u\|_{2,2;[s,t]}\|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]}\|\phi\|_{0,(\infty)} \tag{4.16}$$

as well as

$$\iint_{[s,t]\times\mathbb{R}^d} |c||u|^2|\phi| \,\mathrm{d}r\,\mathrm{d}x \leq \|c\|_{r,q;[s,t]}\|u\|^2_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]}\|\phi\|_{0,(\infty)} . \tag{4.17}$$

This yields the inequality

$$\frac{1}{2}\|\delta\mu_{st}\|_{-1,(\infty)} \leq \mathbf{a}(s,t) + \mathbf{b}(s,t)^{1/(2r)}\mathbf{a}(s,t)^{1/2}\mathbf{u}(s,t)^{\frac{r-1}{2r}} + \mathbf{c}(s,t)^{1/r}\mathbf{u}(s,t)^{\frac{r-1}{r}} , \tag{4.18}$$

as for the case of λ above, the r.h.s. in (4.18) is a control, which proves the first part of (4.9).

Making use again of the bounds (4.14)-(2.17) we obtain finally:

$$\|\delta\mu_{st}\|_{-1,(\infty)} \leq C(T,M,\|b\|_{2r,2q},\|c\|_{r,q},r,q)\|u\|^2_{\mathscr{B}_{s,t}} . \tag{4.19}$$

■

As a straightforward, but important consequence, we have the following result.

Corollary 4.1. *Given a smooth path z and its canonical geometrical lift $\mathbf{Z} \equiv (Z,\mathbb{Z})$, let u be a weak solution of* (4.1)*, in the sense of* (4.2)*. Define the path $u^2 : I \to L^1(\mathbb{R}^d)$ by $u^2_t(x) := u_t(x)^2$, for a.e. $(t,x) \in I\times\mathbb{R}^d$.*

Then, u^2 is a weak solution in the sense of Definition 2.2 to

$$\delta u^2_{st} = 2\int_s^t uAu\,\mathrm{d}r + \hat{B}_{st}\left(u^2_s\right) + \hat{\mathbb{B}}_{st}\left(u^2_s\right) + u^{2,\natural}_{st} , \tag{4.20}$$

on the scale $(W^{k,\infty})_{k\in\mathbb{N}_0}$, where we denote by $\hat{B} \equiv (\hat{B},\hat{\mathbb{B}})$ the $1/\alpha$-unbounded rough driver given by (2.9)*, with ν replaced by $\hat{\nu} := 2\nu$.*

Proof. For simplicity, in this proof we let $\boldsymbol{\sigma} := \sigma^{\cdot} \cdot \nabla \equiv \sum_i \sigma^{\cdot,i}(x)\partial_i$. Define the 2-index distribution-valued map

$$u^{2,\natural}_{st} := \delta u^2_{st} - 2\int_s^t (Au)u \, \mathrm{d}r - \hat{B}_{st}(u^2_s) - \hat{\mathbb{B}}_{st}(u^2_s)\,.$$

Using the equation (4.4) twice we see that for any $\phi \in C^\infty(\mathbb{R}^d)$:

$$\begin{aligned}
\langle u^{2,\natural}_{st}, \phi\rangle &= \int_s^t \left\langle u^2_r - u^2_s, (\boldsymbol{\sigma}^{k,*} + \hat{\nu}^k)\phi\right\rangle \mathrm{d}z^k_r - \langle u^2_s, \hat{\mathbb{B}}^*_{st}\phi\rangle \\
&= \iint_{\Delta^2_{[s,t]}} \left\langle u^2_\tau, (\boldsymbol{\sigma}^{\ell,*} + \hat{\nu}^\ell)(\boldsymbol{\sigma}^{k,*} + \hat{\nu}^k)\phi\right\rangle \mathrm{d}z^\ell_\tau \, \mathrm{d}z^k_r - \langle u^2_s, \hat{\mathbb{B}}^*_{st}\phi\rangle \\
&\qquad + 2\iint_{\Delta^2_{[s,t]}} \langle u_\tau A_\tau u_\tau, (\boldsymbol{\sigma}^{k,*} + \hat{\nu}^k)\phi\rangle \, \mathrm{d}\tau \, \mathrm{d}z^k_r \\
&= \iint_{\Delta^2_{[s,t]}} \left\langle \delta u^2_{s\tau}, (\boldsymbol{\sigma}^{\ell,*} + \hat{\nu}^\ell)(\boldsymbol{\sigma}^{k,*} + \hat{\nu}^k)\phi\right\rangle \mathrm{d}z^\ell_\tau \, \mathrm{d}z^k_r \\
&\qquad + 2\iint_{\Delta^2_{[s,t]}} \langle u_\tau A_\tau u_\tau, (\boldsymbol{\sigma}^{k,*} + \hat{\nu})\phi\rangle \, \mathrm{d}\tau \, \mathrm{d}z^k_r \\
:= \mathcal{T}_{\delta u^2} + \mathcal{T}_A\,.
\end{aligned}$$

From Assumption 2.3 on σ, ν, and the fact that, by the classical theory for (4.2), u belongs to the space $\mathscr{B}$, it is immediately seen that every term above makes sense. It remains to show that each of the terms above belongs to $V^{1-}_{2,\mathrm{loc}}(I;\mathbb{R})$, with a bound depending linearly on $| \! | \phi | \! |_{3,(\infty)}$.

For the first term, observe that

$$\begin{aligned}
\sup_{|\phi|_{W^{1,\infty}} \le 1} \langle \phi, \delta u^2_{st}\rangle &\le | \! | \mu_{st} | \! |_{-1,(\infty)} + \sup_{|\phi|_{W^{1,\infty}} \le 1} \int_s^t \langle \sigma^{ki}\partial_i(u^2), \phi\rangle \, \mathrm{d}z^k \\
&\qquad + \sup_{|\phi|_{W^{1,\infty}} \le 1} \int_s^t \langle \hat{\nu}^k u^2, \phi\rangle \, \mathrm{d}z^k \\
&\le \varepsilon(s,t)\,.
\end{aligned}$$

where $\varepsilon(s,t)$ is a control depending on $|z|_{1-\mathrm{var}}, |\nu|_{L^\infty}, |\sigma|_{W^{1,\infty}}, \sup_{r\in[s,t]} |u_r|^2_{L^2}$ and the control ω_μ given in Lemma 4.1. Consequently, we have the bound

$$\begin{aligned}
\mathcal{T}_{\delta u^2} &\le \Big(\sum\nolimits_{k,\ell} \iint_{\Delta^2_{[s,t]}} \mathrm{d}|z^\ell| \, \mathrm{d}|z^k|\Big) \varepsilon(s,t) | \! | (\boldsymbol{\sigma}^* + \nu)((\boldsymbol{\sigma}^* + \nu))\phi | \! |_{1,(2)} \\
&\le C(|\sigma|_{W^{3,\infty}}, |\nu|_{W^{2,\infty}}) \left(|z|_{1-\mathrm{var};[s,t]}\right)^2 \varepsilon(s,t) | \! | \phi | \! |_{3,(2)}\,.
\end{aligned} \tag{4.21}$$

Similarly, we have

$$\begin{aligned}
\mathcal{T}_A &\le \Big(\sum\nolimits_k \int_s^t \omega_\mu(s,r) \, \mathrm{d}|z^k_r|\Big) | \! | (\boldsymbol{\sigma}^* + \hat{\nu})\phi | \! |_{1,(\infty)} \\
&\le C(|\nu|_{W^{1,\infty}}, |\sigma|_{W^{2,\infty}}) |z|_{1-\mathrm{var};[s,t]} \omega_\mu(s,t)\,.
\end{aligned} \tag{4.22}$$

The conclusion follows from (4.21), (4.22), and Remark 3.3, we have:

$$u^{2,\natural} \in V^{1-}_{2,\mathrm{loc}}(I; (W^{3,\infty})^*)\,,$$

which proves the corollary. ∎

Proof of Proposition 4.1. Testing against $\phi = 1 \in W^{3,\infty}$ in (4.20), we have, using (2.10) and the inequality $|\sum b^i\partial_i u| \leq m/2\sum(b^i)^2 + 1/(2m)\sum(\partial_i u)^2$:

$$\delta\left(|u|^2_{L^2}\right)_{st} + 2m\int_s^t |\nabla u_r|^2_{L^2}\,\mathrm{d}r \leq \iint_{[s,t]\times\mathbb{R}^d}\left(m|\nabla u_r|^2 + \left(\tfrac{1}{m}\textstyle\sum_{i\leq d}(b^i)^2 + 2|c|\right)u^2\right)\mathrm{d}r\,\mathrm{d}x$$
$$+\left(\omega_B(s,t)^\alpha + \omega_B(s,t)^{2\alpha}\right)|u_s|^2_{L^2} + \langle u^{2,\natural}_{st}, 1\rangle\,. \tag{4.23}$$

Note that by (2.14),

$$\begin{aligned}\iint_{[s,t]\times\mathbb{R}^d}\tfrac{1}{m}\big(\textstyle\sum(b^i)^2 + |c|\big)u^2\,\mathrm{d}r\,\mathrm{d}x &\leq \left\|\tfrac{1}{m}\textstyle\sum(b^i)^2 + |c|\right\|_{r,q}\|u\|^2_{\frac{2r}{r-1},\frac{2q}{q-1}}\\ &\leq \left(1+\tfrac{\beta^2}{2m}\right)\left(\|b\|^2_{2r,2q;[s,t]} + \|c\|_{r,q;[s,t]}\right)\|u\|^2_{\mathscr{B}_{s,t}}\\ &= C\left(\mathbf{b}(s,t)^{1/r} + \mathbf{c}(s,t)^{1/r}\right)\|u\|^2_{\mathscr{B}_{s,t}}\,,\end{aligned}$$

where we make use of the notation (4.10) and we recall that $\beta > 0$ denotes the sharpest constant in (2.17). Therefore, defining $G_t := |u_t|^2_{L^2} + \min(1,m)\int_0^t|\nabla u_r|^2_{L^2}\,\mathrm{d}r$ we have

$$\delta G_{st} \leq C\left(\mathbf{b}(s,t)^{1/r} + \mathbf{c}(s,t)^{1/r} + \omega_B(s,t)^\alpha + \omega_B(s,t)^{2\alpha}\right)\|u\|^2_{\mathscr{B}_{s,t}} + \langle u^{2,\natural}_{st}, 1\rangle\,, \tag{4.24}$$

for a constant $C > 0$ depending on m, r, q only. Now, combining Lemma 4.1 and Proposition 3.1, we can estimate the remainder as follows

$$|\!|\!| u^{2,\natural}_{st} |\!|\!|_{-3,(\infty)} \leq C\left(\omega_B(s,t)^{3\alpha} + \omega_B(s,t)^\alpha\right)\|u\|^2_{\mathscr{B}_{s,t}}\,, \tag{4.25}$$

where the constant above depends on $\|b\|_{2r,2q}$, $\|c\|_{r,q}$, but also on $|\sigma|_{W^{3,\infty}}$, $|\nu|_{W^{2,\infty}}$. Hence, using (4.24), (4.9) and (2.14), we obtain that

$$\delta G_{st} \leq \omega(s,t)^{1/\kappa}\Big(\sup_{r\in[s,t]} G_r\Big)\,, \tag{4.26}$$

for the control

$$\omega := C\left(\mathbf{b}^{\kappa/r} + \mathbf{c}^{\kappa/r} + (\omega_Z)^{\kappa\alpha} + (\omega_Z)^{2\kappa\alpha} + (\omega_Z)^{3\kappa\alpha}\right)\,,$$

for an appropriate constant $C = C(\kappa, M, |\sigma|_{W^{2,\infty}}, |\nu|_{W^{1,\infty}})$, where we let $\kappa := \max(1/\alpha, r)$.

Applying Lemma 3.1 with $\varphi := 0$, this gives us the energy inequality (4.3), for a constant depending, through (2.21) and Proposition 3.1, on the quantities

$$\omega_Z, m, M, T, \|b\|_{2r,2q}, \|c\|_{r,q}, |\sigma|_{W^{3,\infty}} \text{ and } |\nu|_{W^{2,\infty}}\,. \qquad \blacksquare$$

5. Tensorization

The aim of this section is to introduce the set-up for the proof of uniqueness presented in Section 6. Recall that in Section 4 we considered a smooth driving signal Z and derived an energy estimate depending only on the rough path norm of the associated canonical lift $\mathbf{Z}$. Nevertheless, the smoothness of Z was only used in Corollary 4.1 in order to verify that u^2 solves (4.20). Accordingly, the result of Proposition 4.1 remains valid in the case of a rough driving signal Z provided one can justify the equation for u^2. This is the main challenge of the proof of uniqueness. Indeed, by linearity of (2.22), uniqueness follows once we show that $\|u\|_{C(I;L^2)} \leq C|u_0|_{L^2}$ is satisfied by every weak solution in the sense of Definition 2.2. However, recall that due to Definition 2.2, the required regularity of test functions that guarantees smallness of the remainder is out of reach for general weak solutions. Consequently, it is not possible to simply test by the solution and to obtain the equation for u^2. Our approach relies on a tensorization procedure which is an analog of the doubling of variables method known from the classical PDE theory.

5.1. **Preliminary material and main result.** For $j = 1, 2$ consider $\mathbf{B}^j \equiv (B^j, \mathbb{B}^j)$, an unbounded rough driver on the scale $(W^{k,2})_{k\in\mathbb{N}_0}$, a drift term $\lambda^j \in V_1^1(I; W^{-1,2})$ and assume the existence of a weak solution $u^j \in C(I; L^2)$ of

$$\mathrm{d}u^j = \mathrm{d}\lambda^j + \mathrm{d}\mathbf{B}^j u^j \,, \tag{5.1}$$

in the sense of Definition 2.2 on the scale $(W^{k,2})_{k\in\mathbb{N}_0}$. For $R > 0$ we define $B_R := \{x \in \mathbb{R}^d : \sum_{i\le d} |x_i|^2 \le R^2\}$ and let

$$\Omega := \left\{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \frac{x - y}{2} \in B_1 \right\} . \tag{5.2}$$

As the first step, we aim to show that the new unknown

$$\mathfrak{u}(x, y) := (u^1 \otimes u^2)(x, y) = u^1(x)u^2(y) \,, \qquad (x, y) \in \Omega \,, \tag{5.3}$$

is itself a solution in the sense of Definition 2.2 of a rough PDE on a suitable scale. This is the first step towards the proof of uniqueness and can be regarded as a linearization of the product operation $u(x)u(x)$. The second step, which we perform in Section 6, then consists of the passage to the diagonal. Namely, we prove that the evolution of $\mathfrak{u}(x, x) = u(x)u(x)$ is given by (2.25).

For $k \in \mathbb{N}_0$, define

$$\mathcal{F}_k := \left\{ \Phi \in W^{k,\infty}(\mathbb{R}^d), \,\mathrm{Supp}\, \Phi \subset \Omega \right\} \,, \quad (\!|\cdot|\!)_k := |\cdot|_{W^{k,\infty}} \,, \tag{5.4}$$

and additionally, let $\mathcal{F}_{-k} := (\mathcal{F}_k)^*$.

Denote by $\mathbf{X} \equiv (X, \mathbb{X})$ the unbounded rough driver given for every $(s, t) \in \Delta$ by

$$\left[\begin{aligned} X_{st} &:= B^1_{st} \otimes \mathrm{id} + \mathrm{id} \otimes B^2_{st}, \\ \mathbb{X}_{st} &:= \mathbb{B}^1_{st} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{B}^2_{st} + B^1_{st} \otimes B^2_{st} \,, \end{aligned} \right. \tag{5.5}$$

(the proof that the properties (RD1)-(RD2) are fulfilled is an easy exercise left to the reader). Furthermore, for every $\Phi \in \mathcal{F}_1$ and $(s, t) \in \Delta$, define the approximate drift as the distribution

$$\pi_{st} := u^1_s \otimes \delta\lambda^2_{st} + \delta\lambda^1_{st} \otimes u^2_s \,. \tag{5.6}$$

Remark 5.1. Let $k \in \mathbb{N}_0$, and define

$$N_k := \#\{\gamma \in \mathbb{N}_0^d, \, |\gamma| := \gamma_1 + \cdots + \gamma_d \le k\} \,.$$

In the proof below, we will make use of the following well-known characterization of the spaces $W^{-k,2} \equiv (W^{k,2})^*$ (see e.g. [Bre10, Proposition 9.20]). For each v in $W^{-k,2}$, there exist a (non-unique) f in $(L^2)^{N_k}$ such that

$$\text{for every } \ \phi \in W^{k,2} \,, \quad {}_{W^{-k,2}}\big\langle v, \phi \big\rangle_{W^{k,2}} = \sum_{|\gamma|\le k} \big(f_\gamma, \mathrm{D}^\gamma \phi \big)_{L^2} \tag{5.7}$$

where $(\cdot, \cdot)_{L^2}$ denotes the L^2 inner product, and $\mathrm{D}^\gamma \phi := \partial_{\gamma_1} \cdots \partial_{\gamma_d} \phi$. Moreover, there holds

$$|v|_{W^{-k,2}} \le |f|_{L^2} \quad \text{and} \quad \inf_{f\in(L^2)^{N_k},\, \text{s.t. (5.7) holds}} \left(\sum_{|\gamma|\le k} |f_\gamma|^2_{L^2} \right)^{1/2} \le |v|_{W^{-k,2}} \,. \tag{5.8}$$

First, we need the following.

Lemma 5.1. *The distribution-valued 2-index map π defined in* (5.6) *has finite variation with respect to $\mathcal{F}_{-1}$, and we have the bound*

$$(\!|\, \pi_{st} \,|\!)_{-1} \leq C \left(\| \delta\lambda^1_{st} \|_{-1,(2)} \| u^2_s \|_{0,(2)} + \| u^1_s \|_{0,(2)} \| \delta\lambda^2_{st} \|_{-1,(2)} \right) , \quad \forall (s,t) \in \Delta \,, \tag{5.9}$$

for some universal constant $C > 0$.

Furthermore, assuming that $\lambda^1, \lambda^2 \in \mathcal{AC}(I; W^{-1,2})$, then there is a unique $\Xi \in V^1_1(I; E)$ such that for every $t \in I$ and every sequence of partitions $|\mathfrak{p}_n| \to 0$ of $[0,t]$ we have

$$\lim_{n\to\infty} \sum_{(\mathfrak{p}_n)} \pi_{t_i t_{i+1}} \to \Xi_t \quad \textit{in} \quad \mathcal{F}_{-1} \,. \tag{5.10}$$

Notation 5.1. For $a \in \mathbb{R}^d$, we will henceforth denote by $\boldsymbol{\tau}_a$ the translation operator, namely for $\psi \in L^2(\mathbb{R}^d)$:

$$\boldsymbol{\tau}_a \psi(x) := \psi(x - a) \,, \quad x \in \mathbb{R}^d \,. \tag{5.11}$$

We recall that $\boldsymbol{\tau}_a$ is an isometry in every L^p space, $p \in [1, \infty]$. In addition, we have the following property: for every p in $[1, \infty)$, and every $f \in L^p$,

$$\| \boldsymbol{\tau}_a f - f \|_{L^p} \to 0 \quad \text{as} \quad a \to 0 \,; \tag{5.12}$$

(it suffices to check this for f in C^∞ and then to argue by density).

Proof of Lemma 5.1. Fix (s,t) in Δ. Due to Remark 5.1, for $j = 1, 2$ there exists $(f^j_\gamma)_{|\gamma|\leq 1}$ in $(L^2)^{N_1}$ such that for every $\phi \in W^{1,2}(\mathbb{R}^d)$:

$${}_{W^{-1,2}(\mathbb{R}^d)}\big\langle \delta\lambda^j_{st}, \phi \big\rangle_{W^{1,2}(\mathbb{R}^d)} = \sum_{|\gamma|\leq 1} \big(\Lambda^j_\gamma, \mathrm{D}^\gamma \phi \big)_{L^2(\mathbb{R}^d)} \,. \tag{5.13}$$

Then, for $\Phi \in \mathcal{F}_1$ we have by definition

$$\begin{aligned}
\langle \pi_{st}, \Phi \rangle &= \int_{\mathbb{R}^d} u^1(x) \, {}_{W^{-1,2}}\big\langle \delta\lambda^2, \Phi(x,\cdot) \big\rangle_{W^{1,2}} \,\mathrm{d}x + \int_{\mathbb{R}^d} u^2(y) \, {}_{W^{-1,2}}\big\langle \delta\lambda^1, \Phi(\cdot,y) \big\rangle_{W^{1,2}} \,\mathrm{d}y \\
&= \sum_{|\gamma|\leq 1} \int_{\mathbb{R}^d} u^1(x) \big(\Lambda^2_\gamma, \mathrm{D}^\gamma_y \Phi(x,\cdot) \big)_{L^2_y(\mathbb{R}^d)} \,\mathrm{d}x + \sum_{|\gamma|\leq 1} \int_{\mathbb{R}^d} \big(\Lambda^1_\gamma, \mathrm{D}^\gamma_x \Phi(\cdot,y) \big)_{L^2_x(\mathbb{R}^d)} u^2(y) \,\mathrm{d}y \\
&\leq C \iint_\Omega \Big(|u^1(x)||\Lambda^2(y)| + |\Lambda^1(x)||u^2(y)| \Big) (|\Phi| + |\nabla_{x,y}\Phi|)(x,y) \,\mathrm{d}x \,\mathrm{d}y \\
&= C \iint_{\mathbb{R}^d \times B_1} \big(|u^1(x_+ + x_-)||\Lambda^2(x_+ - x_-)| + |\Lambda^1(x_+ + x_-)||u^2(x_+ - x_-)| \big) \\
&\qquad\qquad \times (|\Phi| + |\nabla_{x,y}\Phi|)(x_+ + x_-, x_+ - x_-) \,\mathrm{d}x_- \,\mathrm{d}x_+ \\
&\leq C \int_{B_1} \Big(|\boldsymbol{\tau}_{-x_-} u^1|_{L^2_+} |\boldsymbol{\tau}_{x_-} \Lambda^2|_{L^2_+} + |\boldsymbol{\tau}_{-x_-} \Lambda^1|_{L^2_+} |\boldsymbol{\tau}_{x_-} u^2|_{L^2_{x_+}} \Big) \,\mathrm{d}x_- (\!|\, \Phi \,|\!)_1 \\
&= C|B_1| \big(|u^1|_{L^2} |\Lambda^2|_{L^2} + |\Lambda^1|_{L^2} |u^2|_{L^2} \big) (\!|\, \Phi \,|\!)_1 \,,
\end{aligned} \tag{5.14}$$

where in the third line we have made the change of variables $(x_+, x_-) = (\frac{x+y}{2}, \frac{x-y}{2})$. Now, the constant above does not depend on the choice of Λ^1, Λ^2 in (5.13), hence we can take the infimum, which, thanks to (5.8), yields the first part of the Lemma.

We need to justify the existence and uniqueness of Ξ such that (5.10) holds. Recall that since $\lambda^j \in \mathcal{AC}(I; W^{-1,2})$ and since $W^{-1,2}$ is reflexive, then $\dot\lambda_r \equiv \lim_{\epsilon\to 0} (\lambda^j_{r+\epsilon} - \lambda^j_r)/\epsilon \in W^{-1,2}$ exists a.e. in I, and we have

$$\delta\lambda^j_{st} = \int_s^t \dot\lambda^j_r \,\mathrm{d}r$$

(Bochner sense). On the other hand, from similar computations as in (5.14) we have $\int_I \llparenthesis u^1_r \otimes \dot\lambda^2_r + \dot\lambda^1_r \otimes u^2_r \rrparenthesis_{-1}\,\mathrm{d}r < \infty$. Observing that the mapping $r \in I \mapsto f_r := u^1_r \otimes \cdot + \cdot \otimes u^2_r \in L\big(W^{-1,2}; \mathcal{F}_{-1}\big)$ is well defined and continuous, with a norm not exceeding $\|u^1\|_{\infty,2} + \|u^2\|_{\infty,2}$, we can then apply (A.10), so that for every $\mathfrak{p}_n \in \mathscr{P}([0,t])$, $|\mathfrak{p}_n| \to 0$:

$$\sum_{(\mathfrak{p}_n)} \pi_{t_i t_{i+1}} \to \Xi_t \equiv \int_0^t (-u^1_r \otimes \dot\lambda^2_r - \dot\lambda^1_r \otimes u^2_r)\,\mathrm{d}r \quad \text{strongly in } \mathcal{F}_{-1}\,. \qquad \blacksquare$$

The main result of this section is the following.

Proposition 5.1. *(a) There exists $\Pi \in V^1_1(I;\mathcal{F}_{-1})$ such that for every $(s,t) \in \Delta$,*

$$\llparenthesis \delta\Pi_{st} - \pi_{st} \rrparenthesis_{-2} \le \omega(s,t)^a \tag{5.15}$$

for some control ω and some $a > 1$. If in addition λ^1, λ^2 belong to $\mathcal{AC}(I; W^{-1,2})$, then Π is unique and we have $\Pi = \Xi$, where Ξ is as in (5.10)*.*

(b) the tensor product $\mathfrak{u} \equiv u^1 \otimes u^2$ is a weak solution of the rough PDE

$$\mathrm{d}\mathfrak{u} = \mathrm{d}\Pi + \mathrm{d}\mathbf{X}\mathfrak{u}\,, \tag{5.16}$$

on the scale $(\mathcal{F}_k)_{k\in\mathbb{N}_0}$, in the sense of Definition 2.2.

5.2. **Proof of Proposition 5.1.** *Proof of (a).* The first claim follows by the same arguments as in Lemma 5.1, together with an application of the Sewing Lemma (see Appendix A.2). More precisely, there holds for $(s,\theta,t) \in \Delta$:

$$\delta(\pi)_{s\theta t} = -\delta u^1_{s\theta} \otimes \delta\lambda^2_{\theta t} - \delta\lambda^1_{\theta t} \otimes \delta u^2_{s\theta}\,.$$

Now, for $j = 1,2$ let Λ^j in $(L^2)^{N_1}$ such that (5.13) holds (with θ, t instead of s,t), and similarly let $(f^j_\beta)_{|\beta|\le 1} \in (L^2)^{N_1}$ such that for every ϕ in $W^{1,2}$

$$\langle \delta u^j_{s\theta}, \phi \rangle = \sum_{|\beta|\le 1} \big(f^j_\beta, \mathrm{D}^\beta\phi\big)_{L^2}\,. \tag{5.17}$$

Let $\Phi \in \mathcal{F}_2$. Then we see that

$$\begin{aligned}
&\langle \delta\pi_{s\theta t}, \Phi\rangle = \\
&- \sum_{|\gamma|,|\beta|\le 1} \Big(f^1_\beta, \Big(\Lambda^2_\gamma, \mathrm{D}^\beta_x \mathrm{D}^\gamma_y \Phi\Big)_{L^2_y}\Big)_{L^2_x} - \sum_{|\gamma|,|\beta|\le 1} \Big(\Lambda^1_\gamma, \Big(f^2_\beta, \mathrm{D}^\gamma_x \mathrm{D}^\beta_y \Phi\Big)_{L^2_y}\Big)_{L^2_x} \\
&\le C \iint_\Omega \big(|f^1(x)||\Lambda^2(y)| + |\Lambda^1(x)||f^2(y)|\big)\big(|\Phi| + |\nabla_{x,y}\Phi| + |\nabla^2_{x,y}\Phi|\big)(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.
\end{aligned}$$

Proceeding as as before with the change of variables $(x_+, x_-) = (\frac{x+y}{2}, \frac{x-y}{2})$, taking the infimum over $\Lambda^1, \Lambda^2, f^1, f^2$ such that (5.13), (5.17) hold, and then using (5.8), we obtain that

$$\llparenthesis \delta\pi_{s\theta t} \rrparenthesis_{-2} \le C\left(\boldsymbol{|} \delta u^1_{s\theta} \boldsymbol{|}_{-1,(2)} \boldsymbol{|} \delta\lambda^2_{st} \boldsymbol{|}_{-1,(2)} + \boldsymbol{|} \delta\lambda^1_{\theta t} \boldsymbol{|}_{-2,(2)} \boldsymbol{|} \delta u^2_{s\theta} \boldsymbol{|}_{-1,(2)}\right)\,, \quad \forall (s,\theta,t) \in \Delta^2\,.$$

for some universal constant $C > 0$. Hence, for every (s,θ,t) in Δ^2 :

$$\llparenthesis \delta\pi_{s\theta t} \rrparenthesis_{-2} \le C\Big(\omega_{\lambda^1}(s,t)\omega_{\delta u^2}(s,t)^\alpha + \omega_{\delta u^1}(s,t)^\alpha \omega_{\lambda^2}(s,t)\Big) \tag{5.18}$$

where for $j = 1,2$ and $(s,t) \in \Delta$, we set

$$\omega_{\delta u^j}(s,t) := \inf\Big\{\omega(s,t)\,\Big|\,\omega : \Delta_{[s,t]} \to \mathbb{R}_+ \text{ control s.t. } \omega^\alpha \ge \boldsymbol{|} \delta u^j \boldsymbol{|}_{-1,(2)}\Big\}. \tag{5.19}$$

Note that this quantity is finite by Proposition 3.1 and defines a control (this is seen from similar arguments as that of Remark 3.3). Consequently, the r.h.s. of (5.18) fulfills the hypotheses of the Sewing Lemma, i.e. $\delta\pi \in \mathcal{Z}_3^{1-}(I;\mathcal{F}_{-2})$.

Hence by Corollary A.1, there is a unique $\Pi^\dagger$ in $V_1^1(I;\mathcal{F}_{-2})$ such that $(\pi - \delta\Pi^\dagger) \in V_2^{1-}(I;\mathcal{F}_{-2})$. It is given by the rough integral

$$\Pi_t^\dagger = \mathcal{I}_{0t}(\pi) \equiv (\mathcal{F}_{-2}) - \lim_{\substack{|\mathfrak{p}|\to 0\\ \mathfrak{p}\in\mathscr{P}([0,t])}} \sum_{(\mathfrak{p})} \pi_{t_i t_{i+1}}\,. \tag{5.20}$$

We need to justify that $\Pi^\dagger$ can be extended in a unique way to an element Π in $V_1^1(I;\mathcal{F}_{-1})$, which is not trivial since $\mathcal{F}_2$ *is not dense in* $\mathcal{F}_1$. However, letting $|\mathfrak{p}_n| \to 0$ and $\mathcal{I}_n\pi$ be the partial sum associated to $\mathfrak{p}_n$ in the r.h.s. of (5.20), we have that $\limsup_n (\!| \mathcal{I}_n\pi |\!)_{-1} \leq \omega_\pi(s,t) < \infty$, where ω_π is any control such that $\omega_\pi \geq (\!| \pi |\!)_{-1}$. Hence by the Hahn-Banach Theorem, there exists such an extension Π. Finally, by Lemma 5.1, we have $\mathcal{I}_n\pi \to \Xi$ in $\mathcal{F}_{-1}$, yielding that $\Pi = \Xi$. This proves part (a). ∎

Proof of (b). Define $\Pi := \mathcal{I}_{0\cdot}(\pi)$ as above. We have to show that the distribution-valued 2-index map $\mathfrak{u}^\natural$ defined for each $(s,t) \in \Delta$ as

$$\mathfrak{u}^\natural_{st} := \delta\mathfrak{u}_{st} - \delta\Pi_{st} - X_{st}\mathfrak{u}_s - \mathbb{X}_{st}\mathfrak{u}_s\,, \tag{5.21}$$

belongs to $V_{2,\mathrm{loc}}^{1-}(I;\mathcal{F}_{-3})$.

A straightforward, but very useful observation is the following.

Claim 5.1. For $(s,t) \in \Delta$ and $j = 1,2$, we define the corresponding first order remainder

$$u^{j,\sharp}_{st} := \delta u^j_{st} - B^j_{st}u^j_s\,. \tag{5.22}$$

Then we have the identity

$$\mathfrak{u}^\natural_{st} = u^{1,\natural}_{st} \otimes u^2_s + u^1_s \otimes u^{2,\natural}_{st} + \pi_{st} - \delta\Pi_{st} + u^{1,\sharp}_{st} \otimes \delta u^2_{st} + B^1_{st}u^1_s \otimes u^{2,\sharp}_{st}\,. \tag{5.23}$$

Proof of Claim. First observe that adding and subtracting, we have

$$\delta\mathfrak{u}_{st} = \delta u^1_{st} \otimes u^2_s + u^1_s \otimes \delta u^2_{st} + \delta u^1_{st} \otimes \delta u^2_{st}\,,$$

which, omitting time indexes, is equal to :

$$\begin{aligned}
&(\delta u^1 - B^1u^1 - \mathbb{B}^1u^1) \otimes u^2 + u^1 \otimes (\delta u^2 - B^2u^2 - \mathbb{B}^2u^2)\\
&\qquad\qquad + X\mathfrak{u} + \mathbb{X}\mathfrak{u} - B^1u^1 \otimes B^2u^2 + \delta u^1 \otimes \delta u^2\\
&\quad \equiv (\delta u^1 - B^1u^1 - \mathbb{B}^1u^1) \otimes u^2 + u^1 \otimes (\delta u^2 - B^2u^2 - \mathbb{B}^2u^2) + X\mathfrak{u} + \mathbb{X}\mathfrak{u}\\
&\qquad\qquad + (\delta u^1 - B^1u^1) \otimes \delta u^2 + B^1u^1 \otimes (\delta u^2 - B^2u^2)\,.
\end{aligned}$$

Similarly, adding and subtracting the drift term and using (5.21), we obtain that:

$$\begin{aligned}
\mathfrak{u}^\natural_{st} &= (\delta u^1_{st} - B^1_{st}u^1_s - \mathbb{B}^1_{st}u^1_s - \lambda^1_{st}) \otimes u^2_s + u^1_s \otimes (\delta u^2_{st} - B^2_{st}u^2_s - \mathbb{B}^2_{st}u^2_s - \lambda^2_{st})\\
&\quad + \pi_{st} - \delta\Pi_{st} + (\delta u^1_{st} - B^1_{st}u^1_s) \otimes \delta u^2_{st} + B^1_{st}u^1_s \otimes (\delta u^2_{st} - B^2_{st}u^2_s)\,.
\end{aligned}$$

hence the claim is proved. ∎

End of the Proof of Proposition 5.1. Take any Φ in $\mathcal{F}_3$. From the identity (5.23), we can decompose $\langle \mathfrak{u}^\natural, \Phi\rangle$ into

$$\begin{aligned}\langle \mathfrak{u}^\natural_{st}, \Phi\rangle = &\left\langle u^{1,\natural}_{st}\otimes u^2_s + u^1_s\otimes u^{2,\natural}_{st}, \Phi\right\rangle + \langle \pi_{st} - \delta\Pi_{st}, \Phi\rangle \\ &\qquad + \left\langle u^{1,\sharp}_{st}\otimes \delta u^2_{st}, \Phi\right\rangle + \left\langle B^1_{st}u^1_s\otimes u^{2,\sharp}_{st}, \Phi\right\rangle \\ =:& \mathcal{T}_\natural + \mathcal{T}_\lambda + \mathcal{T}^1_\sharp + \mathcal{T}^2_\sharp\,.\end{aligned}$$

In the above formula, it is immediately seen, according to Remark 3.4, that each term above has the needed size in time, namely $\langle \mathfrak{u}^\natural, \Phi\rangle$ belongs to the space $V^{1-}_{2,\mathrm{loc}}(I;\mathbb{R})$. That being said, it is necessary to evaluate $\mathfrak{u}^\natural$ as a path with values in $\mathcal{F}_{-3}$, and not in $\mathscr{D}'(\mathbb{R}^d)$ only. For that purpose, we use the characterization of Sobolev Spaces of negative order given by Remark 5.1. Fix (s,t) in Δ and for $j=1,2$ let $g^j, h^j \in (L^2)^{N_2}$ and $h^j\in(L^2)^{N_3}$ be such that

$$\text{for every } \phi\in W^{2,2}\,,\quad \langle u^{j,\sharp}_{st}, \phi\rangle = \sum\nolimits_{|\gamma|\le 2}\left(g^j_\gamma, \mathrm{D}^\gamma\phi\right)_{L^2} \tag{5.24}$$

$$\text{for every } \phi\in W^{3,2}\,,\quad \langle u^{j,\natural}_{st}, \phi\rangle = \sum\nolimits_{|\beta|\le 3}\left(h^j_\beta, \mathrm{D}^\beta\phi\right)_{L^2}\,, \tag{5.25}$$

and let f^j be as in (5.17).

For the first term, we have by definition:

$$\begin{aligned}\mathcal{T}_\natural =& \left\langle u^2, \langle u^{1,\natural}, \Phi\rangle_x\right\rangle_y + \left\langle u^1, \langle u^{2,\natural}, \Phi\rangle_y\right\rangle_x \\ =& \sum\nolimits_{|\beta|\le 3}\left(u^2, \left(h^1_\beta, \mathrm{D}^\beta_x\Phi\right)_{L^2_x}\right)_{L^2_y} + \sum\nolimits_{|\beta|\le 3}\left(u^1, \left(h^2_\beta, \mathrm{D}^\beta_y\Phi\right)_{L^2_y}\right)_{L^2_x}\,.\end{aligned}$$

Changing variables as before, there comes

$$\begin{aligned}\mathcal{T}_\natural \le& C\iint_{B_1\times\mathbb{R}^d}(|u^2(x_+-x_-)||h^1(x_++x_-)| + |u^1(x_++x_-)||h^2(x_+-x_-)|) \\ &\qquad\qquad \times(|\Phi| + |\nabla\Phi| + |\nabla^2\Phi| + |\nabla^3\Phi|)(x_++x_-, x_+-x_-)\,\mathrm{d}x_+\,\mathrm{d}x_- \\ \le& C\left(|u^2_s|_{L^2}|h^1|_{L^2} + |u^1_s|_{L^2}|h^2|_{L^2}\right)(\!|\,\Phi\,|\!)_3\,,\end{aligned}$$

where again we have used Fubini's theorem, together with the fact that the translations $\boldsymbol{\tau}_{x_-}, \boldsymbol{\tau}_{-x_-}$ are isometries in L^2. Hence, taking the infimum over the choice of h^1, h^2 in (5.25), it holds true that for every $(s,t)\in\Delta$,

$$\mathcal{T}_\natural \le C\left(\|u^2_s\|_{0,(2)}\|u^{2,\natural}_{st}\|_{-3,(2)} + \|u^1_s\|_{0,(2)}\|u^{2,\natural}_{st}\|_{-3,(2)}\right)(\!|\,\Phi\,|\!)_3\,, \tag{5.26}$$

for some constant $C>0$, independent of (s,t) in Δ and Φ in $\mathcal{F}_3$.

For the third term, we have

$$\begin{aligned}\mathcal{T}^1_\sharp =& \left\langle u^{1,\sharp}, \langle\delta u^2, \Phi\rangle_y\right\rangle_x \\ =& \sum_{|\gamma|\le 2, |\beta|\le 1}\left(g^1_\gamma, \left(f^2_\beta, \mathrm{D}^\gamma_x\mathrm{D}^\beta_y\Phi\right)_{L^2_y}\right)_{L^2_x} \le\ C|g^1|_{L^2}|f^2|_{L^2}(\!|\,\Phi\,|\!)_3\,.\end{aligned}$$

Hence, taking the infimum over g^1, f^2 gives

$$\langle\mathcal{T}^1_\sharp, \Phi\rangle \le C\|\delta u^2_{st}\|_{-1,(2)}\|u^{1,\sharp}_{st}\|_{-2,(2)}(\!|\,\Phi\,|\!)_3\,, \tag{5.27}$$

for a constant depending neither on $(s,t)\in\Delta$, neither on Φ in $\mathcal{F}_3$.

Proceeding similarly for the fourth term, there holds:

$$\mathcal{T}_\sharp^2 = \left\langle B^1 u^1, \langle u^{2,\sharp}, \Phi\rangle_y \right\rangle_x = \sum\nolimits_{|\gamma|\le 2} \left(u^1, \left(g^2_\gamma, B^{1,*}_x \mathrm{D}^\gamma_y \Phi\right)_{L^2_y}\right)_{L^2_x}$$

Hence, we have

$$\mathcal{T}_\sharp^2 \le C\omega_{B^1}(s,t)^\alpha \, |\!|\!| u^1_s |\!|\!|_{0,(2)} |\!|\!| u^{2,\sharp}_{st} |\!|\!|_{-2,(2)} (\!| \Phi |\!)_3 \,. \tag{5.28}$$

for some universal constant $C > 0$.

Now, note that the drift term has been already estimated in Lemma 5.1, namely, we have

$$\begin{aligned} \mathcal{T}_\lambda =& \langle (\Lambda\delta\pi)_{st}, \Phi\rangle \\ \le& C\left(\omega_{\lambda^1}(s,t)\omega_{\delta u^2}(s,t)^\alpha + \omega_{\delta u^1}(s,t)^\alpha \omega_{\lambda^2}(s,t)\right) (\!| \Phi |\!)_2 \,. \end{aligned} \tag{5.29}$$

The conclusion follows by (5.26)-(5.27)-(5.28)-(5.29). Indeed, making use of the controls defined in (5.19), and furthermore defining for $j = 1, 2$:

$$\begin{aligned} \omega_{j,\sharp}(s,t) &:= \inf \left\{ \omega(s,t) \;\middle|\; \omega : \Delta_{[s,t]} \to \mathbb{R}_+ \text{ control s.t. } (\omega)^{2\alpha} \ge |\!|\!| u^{j,\sharp} |\!|\!|_{-2,(2)} \right\} \\ \omega_{j,\natural}(s,t) &:= \inf \left\{ \omega(s,t) \;\middle|\; \omega : \Delta_{[s,t]} \to \mathbb{R}_+ \text{ control s.t. } (\omega)^{3\alpha} \ge |\!|\!| u^{j,\natural} |\!|\!|_{-3,(2)} \right\} . \end{aligned}$$

(these quantities are well-defined controls from Proposition 3.1 and Remark 3.3), then we see that:

$$\begin{aligned} (\!| \mathbf{u}^\natural_{st} |\!)_{-3} \le C\left(\alpha, \|u^1\|_{CL^2}, \|u^2\|_{CL^2}\right) \Big(&(\omega_{1,\natural})^{3\alpha} + (\omega_{2,\natural})^{3\alpha} + (\omega_{\delta u^2})^\alpha (\omega_{1,\sharp})^{2\alpha} + (\omega_{B^1})^\alpha (\omega_{2,\sharp})^{2\alpha} \\ &+ \omega_{\lambda^1}(\omega_{\delta u^2})^\alpha + (\omega_{\delta u^1})^\alpha \omega_{\lambda^2} \Big), \end{aligned}$$

where all the controls are evaluated at (s,t). Since each term on the above right hand side is of homogeneity at least 3α, we see that

$$\mathbf{u}^\natural \in V^{1/(3\alpha)}_{2,\mathrm{loc}}(I;\mathcal{F}_{-3}) \subset V^{1-}_{2,\mathrm{loc}}(I;\mathcal{F}_{-3}) \,,$$

which completes the proof of Proposition 5.1. ■

Remark 5.2. Assume that for $j = 1, 2$ and $t \in I$:

$$\lambda^j_t := \int_0^t A^j u^j \,\mathrm{d}r$$

where we are given u^j in $\mathscr{B}$ and

$$A^j(t,x) := \partial_\alpha(a^{j,\alpha\beta}(t,x)\partial_\beta\cdot) + b^{j,\alpha}(t,x)\partial_\alpha + c^j(t,x) \,,$$

with coefficents a^j, b^j, c^j such that Assumption 2.1 and Assumption 2.2 hold. Using the explicit form (4.13) for the control ω_λ appearing in Lemma 4.1, we see that λ^j belongs to $\mathcal{AC}(I;\mathcal{F}_{-1})$. Moreover, making use of the notations of Proposition 5.1, we have for every $t \in I$:

$$\Pi_t := \int_0^t (u^1_r \otimes A^2_r u^2_r + A^1_r u^1_r \otimes u^2_r) \,\mathrm{d}r \,.$$

in the Bochner sense, in $\mathcal{F}_{-1}$.

6. UNIQUENESS

After the preliminary step of tensorization presented in Section 5 we proceed with the proof of uniqueness. The ultimate goal is to test the tensor equation for $u(x)u(y)$ by a Dirac mass $\delta_{x=y}$ which finally gives the desired equation for u^2. To achieve this, we first consider a smooth approximation to the identity ψ_ϵ which is a legal test function for (5.16). The core of the proof then consists in the justification of the passage to the limit as $\epsilon \to 0$. More precisely, it is necessary to bound all the terms in the equation uniformly in $\epsilon \in (0,1)$. Similarly to the a priori estimates in Section 4, the main challenge is to bound the remainder term. Our approach relies on a suitable blow-up transformation together with uniform bounds for all the other terms in the equation which permits to employ again Proposition 3.1 and yields an estimate uniform in ϵ.

Consider $u \in \mathscr{B}$, a weak solution to (2.22) in the sense of Definition 2.2 and define

$$\mathfrak{u}(x,y) := u(x)u(y)\,, \qquad \text{for every } \ (x,y) \ \text{ in } \ \mathbb{R}^d \times \mathbb{R}^d\,. \tag{6.1}$$

Denote by $\mathbf{S} \equiv (S, \mathbb{S})$ the symmetric driver, given for every $(s,t) \in \Delta$, by

$$\left[\begin{aligned} S_{st} &:= B_{st} \otimes \mathrm{id} + \mathrm{id} \otimes B_{st}, \\ \mathbb{S}_{st} &:= \mathbb{B}_{st} \otimes \mathrm{id} + \mathrm{id} \otimes \mathbb{B}_{st} + B_{st} \otimes B_{st}\,, \end{aligned}\right. \tag{6.2}$$

and also by

$$\Pi_t := \int_0^t \Big(A_r u_r \otimes u_r + u_r \otimes A_r u_r \Big)\,\mathrm{d}r\,.$$

Fix $\epsilon > 0$. Then replacing Ω by

$$\Omega_\epsilon := \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \frac{|x-y|}{2} \le \epsilon \right\} \tag{6.3}$$

in Section 5, then Proposition 5.1 and Remark 5.2 yield that

$$\mathrm{d}\mathfrak{u} = \mathrm{d}\Pi + \mathrm{d}\mathbf{S}\mathfrak{u}\,, \tag{6.4}$$

holds with respect to the scale $(\mathcal{F}_k(\Omega_\epsilon))_{k\in\mathbb{N}_0}$, in the sense of Definition 2.2.

We now define the blow-up transformation $T_\epsilon : \mathcal{F}_0(\Omega) \to \mathcal{F}_0(\Omega_\epsilon)$ as follows: given $\Phi \in \mathcal{F}_0(\Omega)$, we let

$$T_\epsilon \Phi(x,y) := (2\epsilon)^{-d} \Phi\left(\frac{x+y}{2} + \frac{x-y}{2\epsilon}, \frac{x+y}{2} - \frac{x-y}{2\epsilon} \right)\,, \quad \text{for any} \quad (x,y) \in \Omega_\epsilon\,. \tag{6.5}$$

This operation is invertible and we have for $(x,y) \in \Omega$:

$$T_\epsilon^{-1}\Phi(x,y) = (2\epsilon)^d \Phi\left(\frac{x+y}{2} + \epsilon\frac{x-y}{2}, \frac{x+y}{2} - \epsilon\frac{x-y}{2} \right)\,. \tag{6.6}$$

Given $k \in \{0,1,2,3\}$ and v in $\mathcal{F}_{-k}(\Omega_\epsilon)$, we can define a distribution $T_\epsilon^* v \in \mathcal{F}_{-k}(\Omega)$ by duality, and similary $T_\epsilon^{-1,*}v$ makes sense as an element of $\in \mathcal{F}_{-k}(\Omega_\epsilon)$.

For any $\Psi \in \mathcal{F}_3(\Omega)$, we can test (6.4) against

$$\Phi := T_\epsilon \Psi \in \mathcal{F}_3(\Omega_\epsilon)\,.$$

We deduce that for all $\Psi \in \mathcal{F}_3(\Omega)$ and $(s,t) \in \Delta$:

$$\langle T_\epsilon^* \delta\mathfrak{u}_{st}, \Psi \rangle = \langle T_\epsilon^* \delta\Pi_{st}, \Psi \rangle + \langle T_\epsilon^* (S_{st} + \mathbb{S}_{st})\mathfrak{u}_s^\epsilon, \Psi \rangle + \langle T_\epsilon^* \mathfrak{u}_{st}^\natural, \Psi \rangle\,,$$

whence letting $\mathfrak{u}^\epsilon := T_\epsilon^*\mathfrak{u}$, $\mathbf{S}^\epsilon := T_\epsilon^* \mathbf{S} T_\epsilon^{-1,*}$, $\Pi^\epsilon := T_\epsilon^*\Pi$, and $\mathfrak{u}^{\natural,\epsilon} := T_\epsilon^*\mathfrak{u}^\natural$, we see that $\mathfrak{u}^\epsilon$ is a weak solution of

$$\mathrm{d}\mathfrak{u}^\epsilon = \mathrm{d}\Pi^\epsilon + \mathrm{d}\mathbf{S}^\epsilon\mathfrak{u}^\epsilon\,, \tag{6.7}$$

with respect to the scale $(\mathcal{F}_k(\Omega))_{k\in\mathbb{N}_0}$, in the sense of Definition 2.2.

As the next step, we establish uniform bounds for the renormalized driver $\mathbf{S}^\epsilon$ as well as for the drift Π^ϵ, which in turn implies a uniform bound for the remainder $u^{\natural,\epsilon}$. The proof of uniqueness is then concluded in Subsection 6.3.

6.1. **Renormalizability of symmetric drivers.** Let us begin with the uniform bound for the driver $\mathbf{S}^\epsilon$. Following [DGHT16a], the following definition will be useful.

Definition 6.1 (Renormalizable drivers)**.** We say that a family $\mathbf{S}^\epsilon \equiv (S^\epsilon, \mathbb{S}^\epsilon)$, $\epsilon \in (0,1)$, of $1/\alpha$-unbounded rough drivers is *renormalizable*, with respect to a scale $(\mathcal{G}_k)$, if there exists a control ω_S such that the bounds (2.7) hold uniformly with respect to $\epsilon \in (0,1)$, namely for all $(s,t) \in \Delta$,

$$|S^\epsilon_{st}|_{L(\mathcal{G}_{-k},\mathcal{G}_{-k-1})} \le \omega_S(s,t)^\alpha\,, \quad \text{for } k=0,1,2 \text{ and} \tag{6.8}$$

$$|\mathbb{S}^\epsilon_{st}|_{L(\mathcal{G}_{-k},\mathcal{G}_{-k-2})} \le \omega_S(s,t)^{2\alpha}\,, \quad \text{for } k=0,1\,. \tag{6.9}$$

For every k we henceforth omit to mention the domain Ω and write $\mathcal{F}_k$ for $\mathcal{F}_k(\Omega)$ (recall (5.2)). We have the following.

Proposition 6.1. *Consider a driver* $\mathbf{S}$ *as in* (6.2) *and define for each* $\epsilon \in (0,1)$:

$$\mathbf{S}^\epsilon \equiv (S^\epsilon, \mathbb{S}^\epsilon) := (T^*_\epsilon S_{st} T^{-1,*}_\epsilon, T^*_\epsilon \mathbb{S}_{st} T^{-1,*}_\epsilon)\,.$$

Then, the family $(\mathbf{S}^\epsilon)_{\epsilon\in(0,1)}$*, is renormalizable with respect to the scale* $(\mathcal{F}_k)$*.*

Moreover, the bounds (6.8)-(6.9) *hold with a control of the form*

$$\omega_S(s,t) := C\left(|\sigma|_{W^{3,\infty}}, |\nu|_{W^{2,\infty}}\right)\omega_Z(s,t)\,, \tag{6.10}$$

where the constant above only depends on the indicated quantities.

We now need to introduce some useful notations.

Notation 6.1. Recall (5.11). Given $a \in \mathbb{R}^d$ and $\epsilon > 0$, it is useful to introduce the "local mean" as the linear map:

$$\boldsymbol{m}^a_\epsilon := \frac{1}{2}\left(\boldsymbol{\tau}_{-\epsilon a} + \boldsymbol{\tau}_{\epsilon a}\right)\,. \tag{6.11}$$

Notation 6.2. For $a \in \mathbb{R}^d$, we define the finite-difference operator

$$\boldsymbol{\Delta}^a_\epsilon := \frac{\boldsymbol{\tau}_{-\epsilon a} - \boldsymbol{\tau}_{\epsilon a}}{2\epsilon}\,. \tag{6.12}$$

For the reader's convenience, the main properties of $\boldsymbol{\Delta}^{x_-}_\epsilon$ are provided in Appendix A.1.

Notation 6.3. Similarly to Section 5, it will be convenient to use the new coordinates $\chi : \Omega \to \mathbb{R}^d \times B_1$ defined by

$$(x_+, x_-) = \chi(x,y) := \left(\tfrac{x+y}{2}, \tfrac{x-y}{2}\right)\,, \quad \text{for } (x,y) \in \Omega\,. \tag{6.13}$$

Note that $|\det \mathrm{D}\chi| = 2^{-d}$ and that $\sqrt{2}\chi$ is a rotation.

Notation 6.4. Given $\Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we will occasionally denote by $\check{\Phi} := \Phi \circ \chi^{-1}$, namely the map $\check{\Phi} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ given by:

$$\check{\Phi}(x_+, x_-) := \Phi(x_+ + x_-, x_+ - x_-)\,, \quad \text{for } (x_+, x_-) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{6.14}$$

Provided $\Phi \in \mathcal{F}_1$, we have the identities

$$\left[\begin{aligned} &[(\nabla_x + \nabla_y)\Phi] \circ \chi^{-1} = \nabla_+ \check{\Phi} \\ &[(\nabla_x - \nabla_y)\Phi] \circ \chi^{-1} = \nabla_- \check{\Phi} \end{aligned}\right. \tag{6.15}$$

where ∇_+, ∇_- denote the gradients with respect to the new variables x_+, x_-. In view of these relations, we will henceforth write (with a slight abuse of notation):

$$\nabla_\pm[\Phi(x,y)] = \nabla_x\Phi(x,y) \pm \nabla_y\Phi(x,y)\,.$$

Proof of Proposition 6.1. By definition we have

$$S^{\epsilon,*}_{st} =: Z^k_{st}T_\epsilon^{-1}(\Gamma^k_x + \Gamma^k_y)T_\epsilon\,.$$

where for $k \le K$, $\Gamma^k : W^{1,\infty}(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is the first order differential operator

$$\Gamma^k = -\sigma^k\cdot\nabla - \operatorname{div}\sigma^k + \nu^k\,. \tag{6.16}$$

Intuitively, the problematic terms are those that contain derivatives. Indeed, whenever we differentiate $T_\epsilon\Phi$, we obtain a blow up in ϵ. The key observation is then that the blow up only appears in the x_- direction and the bad terms are always multiplied by $\sigma(x)-\sigma(y)$ (or similar), which allows to compensate this blow-up by making use of the higher regularity of σ.

Estimate on S^ϵ_{st} in $L(\mathcal{F}_{-0},\mathcal{F}_{-1})$. For any $\Phi\in\mathcal{F}_1$, we have

$$\begin{aligned}
&(\sigma^k(x)\cdot\nabla_x + \sigma^k(y)\cdot\nabla_y)(T_\epsilon\Phi)(x,y)\\
&\qquad= \sigma^k(x)\cdot T_\epsilon\Big(\tfrac12\nabla_+\Phi + \tfrac{1}{2\epsilon}\nabla_-\Phi\Big) + \sigma^k(y)\cdot T_\epsilon\Big(\tfrac12\nabla_+\Phi - \tfrac{1}{2\epsilon}\nabla_-\Phi\Big)\\
&\qquad= \Big(\frac{\sigma^k(x)+\sigma^k(y)}{2}\Big)\cdot T_\epsilon\nabla_+\Phi + \Big(\frac{\sigma^k(x)-\sigma^k(y)}{2\epsilon}\Big)\cdot T_\epsilon\nabla_-\Phi.
\end{aligned}\tag{6.17}$$

Now, making use of the notations (6.11) and (6.12) we obtain that for a.e. x, y in $\mathbb{R}^d\times\mathbb{R}^d$

$$T_\epsilon^{-1}(\Gamma^k_x+\Gamma^k_y)T_\epsilon \equiv -\left(\boldsymbol{m}^{x_-}_\epsilon\sigma^k\right)(x_+)\cdot\nabla_+ - \left(\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k\right)(x_+)\cdot\nabla_- + 2\boldsymbol{m}^{x_-}_\epsilon(-\operatorname{div}\sigma^k+\nu^k) \tag{6.18}$$

and we abbreviate

$$x_+ := \frac{x+y}{2}\,,\quad x_- := \frac{x-y}{2}\,. \tag{6.19}$$

For the first term in (6.18), we have

$$\begin{aligned}
(\!|\left(\boldsymbol{m}^{x_-}_\epsilon\sigma^k\right)\cdot\nabla_+\Phi|\!)_0 &\equiv \operatorname*{ess\,sup}_{x_+,x_-}\left|\Big(\frac{\boldsymbol{\tau}_{-\epsilon x_-}+\boldsymbol{\tau}_{\epsilon x_-}}{2}\Big)\sigma^k(x_+)\cdot\nabla_+\check{\Phi}(x_+,x_-)\right|\\
&\le |\sigma|_{L^\infty}(\!|\Phi|\!)_1\,.
\end{aligned}$$

For the second term, using Lemma A.1 and the fact that a.e., $\operatorname{Supp}\check{\Phi}(x_+,\cdot)\subset B_1$ we have

$$(\!|(\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k)\cdot\nabla_-\Phi|\!)_0 \le |\nabla\sigma|_{L^\infty}(\!|\Phi|\!)_1\,. \tag{6.20}$$

Concerning the last term in (6.18), we have

$$\begin{aligned}
(\!|2\boldsymbol{m}^{x_-}_\epsilon(-\operatorname{div}\sigma^k+\nu^k)\Phi|\!)_0 &\le \operatorname*{ess\,sup}_{x_+,x_-}\left|\big(\boldsymbol{\tau}_{-\epsilon x_-}+\boldsymbol{\tau}_{\epsilon x_-}\big)(\nu^k-\operatorname{div}\sigma^k)(x_+)\check{\Phi}(x_+,x_-)\right|\\
&\le 2(|\nu|_{L^\infty}+|\operatorname{div}\sigma|_{L^\infty})(\!|\Phi|\!)_0\,.
\end{aligned}$$

Summing these bounds, we obtain the first estimate, namely:

$$|S^{\epsilon,*}_{st}|_{L(\mathcal{F}_1,\mathcal{F}_0)} \le C(|\sigma|_{W^{1,\infty}},|\nu|_{L^\infty})\omega_Z(s,t)^\alpha\,. \tag{6.21}$$

Estimate on S^ϵ_{st} in $L(\mathcal{F}_{-2},\mathcal{F}_{-3})$. Let $\Phi\in\mathcal{F}_3$. First we observe that since the change of coordinates $\sqrt{2}\chi$ is a rotation, in order to estimate $(\!|S^{\epsilon,*}_{st}\Phi|\!)_2$, it is sufficient to estimate

$\llparenthesis (\nabla_\pm)^2 S^{\epsilon,*}_{st}\Phi \rrparenthesis_0$. To this end, we further note that the only critical term in (6.18) is the second one which contains ϵ^{-1}. But in that case, it holds

$$\begin{aligned}\nabla_-[(\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k)(x_+)\cdot\nabla_-\Phi] &= \boldsymbol{m}^{x_-}_\epsilon(\nabla\sigma^k)(x_+)\cdot\nabla_-\Phi + (\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k)(x_+)\cdot\nabla_-^2\Phi,\\ \nabla_+[(\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k)(x_+)\cdot\nabla_-\Phi] &= \boldsymbol{\Delta}^{x_-}_\epsilon(\nabla\sigma^k)(x_+)\cdot\nabla_-\Phi + (\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k)(x_+)\cdot\nabla_+\nabla_-\Phi,\end{aligned}\tag{6.22}$$

where, similarly as before, Lemma A.1 yields that a.e. on Ω :

$$|\boldsymbol{\Delta}^{x_-}_\epsilon(\nabla\sigma^k)|\leq|\sigma|_{W^{2,\infty}},\qquad|\boldsymbol{\Delta}^{x_-}_\epsilon\sigma^k|\leq|\sigma|_{W^{1,\infty}}\,.$$

By the same arguments we can proceed further and apply $\nabla_\pm$ to (6.22). This finally leads to

$$|S^{\epsilon,*}_{st}|_{L(\mathcal{F}_3,\mathcal{F}_2)}\leq C(|\sigma|_{W^{3,\infty}},|\nu|_{W^{2,\infty}})\omega_Z(s,t)^\alpha\,.$$

Estimates on $\mathbb{S}^\epsilon$ in $L(\mathcal{F}_{-0},\mathcal{F}_{-2})$ and $L(\mathcal{F}_{-1},\mathcal{F}_{-3})$. Using geometricity, renormalizability of the term $\mathbb{S}^\epsilon$ can be reduced to the previous cases. This is a consequence of the identity

$$\mathbb{S}^*_{st}\overset{\text{def}}{=}\mathbb{Z}^{k\ell}_{st}(\Gamma^\ell_x\Gamma^k_x+\Gamma^\ell_y\Gamma^k_y)+Z^k_{st}Z^\ell_{st}\Gamma^\ell_y\Gamma^k_x=\mathbb{Z}^{k\ell}_{st}(\Gamma^\ell_x+\Gamma^\ell_y)(\Gamma^k_x+\Gamma^k_y)\,,\qquad(s,t)\in\Delta\,,\tag{6.23}$$

where Γ^k is as in (6.16).

Indeed, emphasizing summations, denoting by $\mathrm{sym}\mathbb{Z}^{k\ell}_{st}:=\frac12(\mathbb{Z}^{k\ell}_{st}+\mathbb{Z}^{\ell k}_{st})\equiv\frac12Z^k_{st}Z^\ell_{st}$ and $\mathrm{anti}\mathbb{Z}^{k\ell}_{st}:=\frac12(\mathbb{Z}^{k\ell}_{st}-\mathbb{Z}^{\ell k}_{st})$, and splitting the term $\sum_{k,\ell}Z^k_{st}Z^\ell_{st}\Gamma^\ell_y\Gamma^k_x$ into two equal parts, one can write:

$$\begin{aligned}\mathbb{S}^*&=\sum_{k,\ell}(\mathrm{sym}\mathbb{Z}^{k\ell}_{st}+\mathrm{anti}\mathbb{Z}^{k\ell}_{st})(\Gamma^\ell_x\Gamma^k_x+\Gamma^\ell_y\Gamma^k_y)+\sum_{k,\ell}\frac{Z^k_{st}Z^\ell_{st}}{2}\Gamma^\ell_y\Gamma^k_x+\sum_{k',\ell'}\frac{Z^{\ell'}_{st}Z^{k'}_{st}}{2}\Gamma^{\ell'}_x\Gamma^{k'}_y\\&=\sum_{k,\ell}\mathrm{sym}\mathbb{Z}^{k\ell}_{st}(\Gamma^\ell_x+\Gamma^\ell_y)(\Gamma^k_x+\Gamma^k_y)+\sum_{k,\ell}\mathrm{anti}\mathbb{Z}^{k\ell}_{st}(\Gamma^\ell_x\Gamma^k_x+\Gamma^\ell_y\Gamma^k_y)\,.\end{aligned}\tag{6.24}$$

However, using antisymmetry, the second term above can be written as $\sum_{k,\ell}\mathrm{anti}\mathbb{Z}^{k\ell}_{st}(\Gamma^\ell_x+\Gamma^\ell_y)(\Gamma^k_x+\Gamma^k_y)$. Summing in (6.24), we see that (6.23) holds.

Now, let Φ in $\mathcal{F}_2$ and estimate

$$\begin{aligned}\llparenthesis\mathbb{S}^{\epsilon,*}_{st}\Phi\rrparenthesis_0&\leq|\mathbb{Z}^{k\ell}_{st}|\llparenthesis T_\epsilon^{-1}(\Gamma^\ell_x+\Gamma^\ell_y)T_\epsilon T_\epsilon^{-1}(\Gamma^k_x+\Gamma^k_y)T_\epsilon\Phi\rrparenthesis_0\\&\leq C(|\sigma|_{W^{1,\infty}},|\nu|_{L^\infty})\omega_Z(s,t)^{2\alpha}\llparenthesis T_\epsilon^{-1}(\Gamma^k_x+\Gamma^k_y)T_\epsilon\Phi\rrparenthesis_1\\&\leq C(|\sigma|_{W^{2,\infty}},|\nu|_{W^{1,\infty}})\omega_Z(s,t)^{2\alpha}\llparenthesis\Phi\rrparenthesis_2\,,\end{aligned}$$

where we have used the bounds obtained in the first part. This yields our first estimate.

The second estimate again reduces to the previous bounds: we have for $\Phi\in\mathcal{F}_3$:

$$\begin{aligned}\llparenthesis\mathbb{S}^{\epsilon,*}_{st}\Phi\rrparenthesis_1&\leq|\mathbb{Z}^{k\ell}_{st}|\llparenthesis T_\epsilon^{-1}(\Gamma^\ell_x+\Gamma^\ell_y)T_\epsilon T_\epsilon^{-1}(\Gamma^k_x+\Gamma^k_y)T_\epsilon\Phi\rrparenthesis_1\\&\leq C(|\sigma|_{W^{2,\infty}},|\nu|_{W^{1,\infty}})\omega_Z(s,t)^{2\alpha}\llparenthesis T_\epsilon^{-1}(\Gamma^k_x+\Gamma^k_y)T_\epsilon\Phi\rrparenthesis_2\\&\leq C(|\sigma|_{W^{3,\infty}},|\nu|_{W^{2,\infty}})\omega_Z(s,t)^{2\alpha}\llparenthesis\Phi\rrparenthesis_3\,,\end{aligned}$$

which proves the claimed bound. ■

6.2. **Uniform bound on the drift.** We proceed with a uniform estimate for the drift in (6.7).

Proposition 6.2. *There exists a control ω_Π, depending on u in $\mathscr{B}$, b in $L^{2r}L^{2q}$, c in L^rL^q, and on M,r,q, such that uniformly in $\epsilon\in(0,1)$, for every (s,t) in Δ :*

$$\llparenthesis\delta\Pi^\epsilon_{st}\rrparenthesis_{-1}\leq\omega_\Pi(s,t)\,.\tag{6.25}$$

Furthermore, we have the bound

$$\omega_{\Pi}(s,t) \le C(M,r,q)\Big(\|u\|_{\infty,2;[s,t]}\|\nabla u\|_{1,2;[s,t]} + \|\nabla u\|^2_{2,2;[s,t]} + \|b\|_{2r,2q;[s,t]}\|u\|^2_{\mathscr{B}_{s,t}} + \|c\|_{r,q;[s,t]}\|u\|^2_{\mathscr{B}_{s,t}}\Big)\,, \tag{6.26}$$

uniformly over $(s,t)\in\Delta$, *where* $C>0$ *depends only the listed quantities.*

Let $k\ge 0$, and assume that we are given a measurable $v(x,y)$ in $\mathcal{F}_{-k}(\Omega_\epsilon)$, such that its trace $\boldsymbol{\gamma}_\Gamma v$ onto the diagonal $\Gamma := \{x,y\in\mathbb{R}^{2d} : x=y\}$ is a well-defined element in $(W^{k,\infty}(\Gamma))^*$ (this is the case for instance if $v(x,y)=f^1(x)f^2(y)$ where $f^1\in W^{-k,2}(\mathbb{R}^d)$ and $f^2\in W^{k,2}(\mathbb{R}^d)$). The adjoint of T_ϵ is given a.e. on Ω by the formula

$$T^*_\epsilon v(x,y) = 2^{-d}\left(\boldsymbol{\tau}_{-\epsilon\frac{x-y}{2}}\otimes\boldsymbol{\tau}_{\epsilon\frac{x-y}{2}}\right)v\Big(\frac{x+y}{2},\frac{x+y}{2}\Big)\,,\quad (x,y)\in\Omega\,, \tag{6.27}$$

which, integrating against $\Phi\in\mathcal{F}_k$, and letting $(x,+,x_-) := \chi(x,y)$ yields the representation

$$\begin{aligned}\langle T^*_\epsilon v,\Phi\rangle &= \iint_{\mathbb{R}^d\times B_1}\left(\boldsymbol{\tau}_{-\epsilon x_-}\otimes\boldsymbol{\tau}_{\epsilon x_-}\right)v(x_+,x_+)\Phi(x_++x_-,x_+-x_-)\,\mathrm{d}x_+\,\mathrm{d}x_-\\ &= \int_{B_1}{}_{W^{k,\infty}(\mathbb{R}^d)^*}\big\langle\boldsymbol{\gamma}_\Gamma\left(\boldsymbol{\tau}_{-\epsilon x_-}\otimes\boldsymbol{\tau}_{\epsilon x_-}\right)v,\check{\Phi}(\cdot,x_-)\big\rangle_{W^{k,\infty}(\mathbb{R}^d)}\,\mathrm{d}x_-\,.\end{aligned} \tag{6.28}$$

Proof. By definition, we have $\delta\Pi^\epsilon_{st} = \int_s^t\langle Au\otimes u+u\otimes Au, T_\epsilon\Phi\rangle\,\mathrm{d}r$. For notational simplicity, we now fix r in $[s,t]$, and denote by $u:=u_r$, $a^{ij}=a^{ij}_r$. For $\Phi\in\mathcal{F}_1$ we have

$$\begin{aligned}&\langle Au\otimes u+u\otimes Au, T_\epsilon\Phi\rangle\\ &\quad= {}_{\mathcal{F}_{-1}(\Omega_\epsilon)}\big\langle\operatorname{div}_x(a_x\nabla_x u_x)u_y, T_\epsilon\Phi\big\rangle_{\mathcal{F}_1(\Omega_\epsilon)} + {}_{\mathcal{F}_{-1}(\Omega_\epsilon)}\big\langle\operatorname{div}_y(a_y\nabla_y u_y)u_x, T_\epsilon\Phi\big\rangle_{\mathcal{F}_1(\Omega_\epsilon)}\\ &\qquad+\iint_{\Omega_\epsilon}b^i(x)\partial_i u(x)u(y)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y+\iint_{\Omega_\epsilon}b^i(y)\partial_i u(y)u(x)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad+\iint_{\Omega_\epsilon}c(x)u(x)u(y)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y+\iint_{\Omega_\epsilon}c(y)u(x)u(y)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y\,,\\ &\quad=: \mathcal{T}^1_a+\mathcal{T}^2_a+\mathcal{T}^1_b+\mathcal{T}^2_b+\mathcal{T}^1_c+\mathcal{T}^2_c\,.\end{aligned} \tag{6.29}$$

Estimate on $\mathcal{T}_a$. Using (6.28), the first term can be written as:

$$\begin{aligned}\mathcal{T}^1_a &= \int_{B_1}{}_{(W^{1,\infty}(\mathbb{R}^d))^*}\Big\langle\boldsymbol{\gamma}_\Gamma\big[\boldsymbol{\tau}_{-\epsilon x_-}\operatorname{div}(a\nabla u)\boldsymbol{\tau}_{\epsilon x_-}u\big],\check{\Phi}(\cdot,x_-)\Big\rangle_{W^{1,\infty}(\mathbb{R}^d)}\mathrm{d}x_-\\ &= \int_{B_1}{}_{(W^{1,\infty}(\mathbb{R}^d))^*}\Big\langle\boldsymbol{\gamma}_\Gamma\big[\operatorname{div}_{x_+}\big(\boldsymbol{\tau}_{-\epsilon x_-}(a\nabla u)\big)\boldsymbol{\tau}_{\epsilon x_-}u\big],\check{\Phi}(\cdot,x_-)\Big\rangle_{W^{1,\infty}(\mathbb{R}^d)}\mathrm{d}x_-\\ &= \int_{B_1}{}_{W^{-1,2}_+}\Big\langle\operatorname{div}_{x_+}\big(\boldsymbol{\tau}_{-\epsilon x_-}(a\nabla u)\big),\boldsymbol{\tau}_{\epsilon x_-}u\check{\Phi}(\cdot,x_-)\Big\rangle_{W^{1,2}_+}\mathrm{d}x_-\\ &= -\int_{B_1}\Big(\boldsymbol{\tau}_{-\epsilon x_-}[a\nabla u],\nabla_+\big(\boldsymbol{\tau}_{\epsilon x_-}u(x_+)\big)\check{\Phi}(\cdot,x_-)+\boldsymbol{\tau}_{\epsilon x_-}u\nabla_+\check{\Phi}(\cdot,x_-)\Big)_{L^2_+}\mathrm{d}x_-\\ &= -\int_{B_1}\Big(\boldsymbol{\tau}_{-\epsilon x_-}[a\nabla u],\boldsymbol{\tau}_{\epsilon x_-}\nabla u\check{\Phi}(\cdot,x_-)\Big)_{L^2_+}\mathrm{d}x_-\\ &\qquad-\int_{B_1}\Big(\boldsymbol{\tau}_{-\epsilon x_-}[a\nabla u],\boldsymbol{\tau}_{\epsilon x_-}u\nabla_+\check{\Phi}(\cdot,x_-)\Big)_{L^2_+}\mathrm{d}x_-\,.\end{aligned} \tag{6.30}$$

Using that $\boldsymbol{\tau}_{\epsilon x_-}$ leaves the L^2 norm invariant for every fixed x_- in $\mathbb{R}^d$, we have

$$\begin{aligned}\mathcal{T}_a^1 &\leq \int_{B_1} |\boldsymbol{\tau}_{-\epsilon x_-}[a\nabla u]|_{L^2_+} |\boldsymbol{\tau}_{\epsilon x_-}\nabla u|_{L^2_+} |\check{\Phi}(\cdot, x_-)|_{L^\infty_+} \,\mathrm{d}x_- \\ &\qquad + \int_{B_1} |\boldsymbol{\tau}_{-\epsilon x_-}[a\nabla u]|_{L^2_+} |\boldsymbol{\tau}_{\epsilon x_-} u|_{L^2_+} |\nabla_+ \check{\Phi}(\cdot, x_-)|_{L^\infty_+} \,\mathrm{d}x_- \\ &= \int_{B_1} |a\nabla u|_{L^2_+} |\nabla u|_{L^2_+} |\check{\Phi}(\cdot, x_-)|_{L^\infty_+} \,\mathrm{d}x_- + \int_{B_1} |a\nabla u|_{L^2_+} |u|_{L^2_+} |\nabla_+ \check{\Phi}(\cdot, x_-)|_{L^\infty_+} \,\mathrm{d}x_- \,,\end{aligned}$$

Hence, doing similar computations for $\mathcal{T}_a^2$, it follows that

$$\int_s^t \mathcal{T}_a \,\mathrm{d}r \leq 2M \left(\|\nabla u\|_{2,2}^2 (\!| \Phi |\!)_0 + \|\nabla u\|_{2,2} \|u\|_{\infty,2} (\!| \Phi |\!)_1 \right) . \tag{6.31}$$

Estimate on $\mathcal{T}_b$. By (6.27), we have

$$\begin{aligned}\mathcal{T}_b^1 &= \iint_{B_1\times\mathbb{R}^d} \boldsymbol{\tau}_{-\epsilon x_-}(b^i \partial_i u)(x_+) \boldsymbol{\tau}_{\epsilon x_-} u(x_+) \check{\Phi}(x_+, x_-) \,\mathrm{d}x_+ \,\mathrm{d}x_- \\ &\leq \int_{B_1} |\boldsymbol{\tau}_{-\epsilon x_-} b|_{L^{2q}_+} |\boldsymbol{\tau}_{-\epsilon x_-}\nabla u|_{L^2_+} |\boldsymbol{\tau}_{\epsilon x_-} u|_{L_+^{\frac{2q}{q-1}}} \,\mathrm{d}x_- (\!| \Phi |\!)_0 \,.\end{aligned} \tag{6.32}$$

Using Hölder, (2.17), and then proceeding similarly for $\mathcal{T}_b^2$, we obtain

$$\begin{aligned}\int_s^t (\mathcal{T}_b^1 + \mathcal{T}_b^2) \,\mathrm{d}r &\leq 2\|b\|_{2r,2q;[s,t]} \|\nabla u\|_{2,2;[s,t]} \|u\|_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]} (\!| \Phi |\!)_0 \\ &\leq 2\beta \|b\|_{2r,2q;[s,t]} \|\nabla u\|_{2,2;[s,t]} \|u\|_{\mathscr{B}_{s,t}} (\!| \Phi |\!)_0 \,.\end{aligned} \tag{6.33}$$

Estimate on $\mathcal{T}_c$. Similarly, it suffices to show the estimate for $\mathcal{T}_c^1$. Using again (6.27), there comes

$$\mathcal{T}_c^1 = \iint_{B_1\times\mathbb{R}^d} \boldsymbol{\tau}_{-\epsilon x_-}[cu](x_+) \boldsymbol{\tau}_{\epsilon x_-} u(x_+) \check{\Phi} \,\mathrm{d}x_+ \,\mathrm{d}x_- \,. \tag{6.34}$$

Hence, Hölder inequality and (2.17) yield

$$\int_s^t (\mathcal{T}_c^1 + \mathcal{T}_c^2) \,\mathrm{d}r \leq 2\|c\|_{r,q;[s,t]} \|u\|^2_{\frac{2r}{r-1},\frac{2q}{q-1};[s,t]} \leq 2\beta^2 \|c\|_{r,q;[s,t]} \|u\|^2_{\mathscr{B}_{s,t}} (\!| \Phi |\!)_0 \,. \tag{6.35}$$

Combining (6.31), (6.33) and (6.35), we obtain the claimed bound. ■

6.3. **The proof of uniqueness.** Finally, we have all in hand to complete the proof of uniqueness. To this end, we let $\omega_\Pi(s,t)$ be as in Proposition 6.2 and recall that according to Proposition 3.1, the following uniform estimate holds true for the remainder term:

$$(\!| \mathfrak{u}^{\natural,\epsilon}_{st} |\!)_{-3} \leq C\Big(\sup_{r\in[s,t]} (\!| \mathfrak{u}^\epsilon_r |\!)_{-0} \omega_B(s,t)^{3\alpha} + \omega_\Pi(s,t)\omega_B(s,t)^\alpha \Big). \tag{6.36}$$

for $(s,t)\in\Delta$. Note that for every $u^1, u^2 \in L^2$ we have

$$\begin{aligned}&\iint_{B_1\times\mathbb{R}^d} |T^*_\epsilon(u^1\otimes u^2)(x,y)| \,\mathrm{d}x \,\mathrm{d}y \\ &\qquad = \iint_{B_1\times\mathbb{R}^d} |\boldsymbol{\tau}_{-\epsilon x_-} u^1(x_+) \boldsymbol{\tau}_{\epsilon x_-} u^2(x_+)| \,\mathrm{d}x_+ \,\mathrm{d}x_- \leq C|u^1|_{L^2} |u^2|_{L^2} \,.\end{aligned} \tag{6.37}$$

Since we have the embedding $L^1(\Omega) \subset L^\infty(\Omega)^*$, using (6.37) with $u^1 = u^2 = u$, there comes

$$\sup_{r\in[s,t]} (\!| \mathfrak{u}^\epsilon_r |\!)_{-0} \leq C \sup_{r\in[s,t]} |u_r|^2_{L^2} \,, \tag{6.38}$$

uniformly in $\epsilon > 0$. Combining the latter with (6.36) yields therefore a uniform bound of the remainder $\mathfrak{u}^{\natural,\epsilon}$.

Now, take $\phi \in W^{3,\infty}(\mathbb{R}^d)$ and $\psi \in C_c^\infty(B_1)$ with $\int_{B_1} \psi \,\mathrm{d}x = 1$ and define

$$\Phi(x,y) := \phi\left(\tfrac{x+y}{2}\right)\psi\left(\tfrac{x-y}{2}\right) . \tag{6.39}$$

Observe furthermore that $(\!|\,\Phi\,|\!)_3 \leq C|\phi|_{W^{3,\infty}} \equiv C\|\phi\|_{3,(\infty)}$ for a positive constant depending on ψ only.

Lemma 6.1. *Let $u_t^2(x) := u_t(x)^2$ which defines an element of*

$$C(I; L^1(\mathbb{R}^d)) \subset C(I; (L^\infty(\mathbb{R}^d))^*).$$

Then we have for every ϕ in $W^{3,\infty}$:

$$\langle \delta u^2_{st}, \phi\rangle = \int_s^t \big(-2\langle a^{ij}\partial_j u, \partial_i(u\phi)\rangle + \langle b^i\partial_i(u^2) + 2cu^2, \phi\rangle\big)\,\mathrm{d}r \\ + \langle u_s^2, \hat{B}^*_{st}\phi\rangle + \langle u_s^2, \hat{\mathbb{B}}^*_{st}\phi\rangle + \langle u^{2,\natural}_{st}, \phi\rangle\,, \tag{6.40}$$

where $\hat{\mathbf{B}} \equiv (\hat{B}, \hat{\mathbb{B}})$ is obtained by replacing ν by 2ν in the definition of $\mathbf{B}$, and $u^{2,\natural}$ belongs to $V^{1-}_{2,\mathrm{loc}}(I, (W^{3,\infty})^*)$. Moreover the latter remainder term is estimated by the right hand side of (6.36).

Proof. Recall that, by definition of $\mathfrak{u}^{\natural,\epsilon}$:

$$\langle \delta\mathfrak{u}_{st}, T_\epsilon\Phi\rangle = \langle \delta\Pi_{st}, T_\epsilon\Phi\rangle + \langle S_{st}\mathfrak{u}_s, T_\epsilon\Phi\rangle + \langle \mathbb{S}_{st}\mathfrak{u}_s, T_\epsilon\Phi\rangle + \langle \mathfrak{u}^{\natural,\epsilon}_{st}, \Phi\rangle\,,$$

where, gathering the terms in (6.31), (6.33), (6.34), it holds:

$$\begin{aligned}\langle \delta\Pi_{st}, T_\epsilon\Phi\rangle = \iiint_{[s,t]\times B_1\times\mathbb{R}^d} \Big[&- \boldsymbol{\tau}_{-\epsilon x_-}(a^{ij}\partial_j u)(\boldsymbol{\tau}_{\epsilon x_-}\partial_i u)(x_+)\phi(x_+)\psi(x_-)\\ &- \boldsymbol{\tau}_{-\epsilon x_-}(a^{ij}\partial_j u)(\boldsymbol{\tau}_{\epsilon x_-}u)(x_+)\partial_i\phi(x_+)\psi(x_-)\\ &- \boldsymbol{\tau}_{\epsilon x_-}(a^{ij}\partial_j u)(\boldsymbol{\tau}_{-\epsilon x_-}\partial_i u)(x_+)\phi(x_+)\psi(x_-)\\ &- \boldsymbol{\tau}_{\epsilon x_-}(a^{ij}\partial_j u)(\boldsymbol{\tau}_{-\epsilon x_-}u)(x_+)\partial_i\phi(x_+)\psi(x_-)\\ &+ \boldsymbol{\tau}_{-\epsilon x_-}(b^i\partial_i u)(x_+)\boldsymbol{\tau}_{\epsilon x_-}u(x_+)\phi(x_+)\psi(x_-)\\ &+ \boldsymbol{\tau}_{\epsilon x_-}(b^i\partial_i u)(x_+)\boldsymbol{\tau}_{-\epsilon x_-}u(x_+)\phi(x_+)\psi(x_-)\\ &+ \boldsymbol{\tau}_{-\epsilon x_-}(cu)(x_+)\boldsymbol{\tau}_{\epsilon x_-}u(x_+)\phi(x_+)\psi(x_-)\\ &+ \boldsymbol{\tau}_{\epsilon x_-}(cu)(x_+)\boldsymbol{\tau}_{-\epsilon x_-}u(x_+)\phi(x_+)\psi(x_-)\Big]\,\mathrm{d}x_+\,\mathrm{d}x_-\,\mathrm{d}r\\ =: \sum_{i=1}^8 \mathcal{T}^i\,. \qquad\qquad &\end{aligned}$$

Step 1: convergence of the drift. Property (5.12), Assumption 2.1 and the dominated convergence theorem imply

$$\begin{aligned}\mathcal{T}^1 + \mathcal{T}^3 \to -2\int_{B_1}\psi(x_-)\Big(\iint_{[s,t]\times\mathbb{R}^d} a^{ij}(x_+)\partial_j u(x_+)\partial_i u(x_+)\phi(x_+)\,\mathrm{d}x_+\,\mathrm{d}r\Big)\,\mathrm{d}x_-\\ \equiv -2\int_s^t \langle a^{ij}\partial_j u, \partial_i u\phi\rangle\,\mathrm{d}r\,,\end{aligned}$$

since $\int_{B_1}\psi\,\mathrm{d}x_- = 1$. Likewise, it holds $\mathcal{T}^2 + \mathcal{T}^4 \to -2\int_s^t \langle a^{ij}\partial_j u, u\partial_i\phi\rangle\,\mathrm{d}r$.

Now, because of (5.12), it follows that

$$\mathcal{T}^5+\mathcal{T}^6\to 2\int_s^t\langle b^i\partial_i u,u\phi\rangle\,\mathrm{d}r\,,\quad\text{and}\quad \mathcal{T}^7+\mathcal{T}^8\to 2\int_s^t\langle cu^2,\phi\rangle\,\mathrm{d}r\,.$$

Summing all the terms above, we end up with the claimed convergence.

Step 2: Convergence of the left hand side. We have:

$$\begin{aligned}
\langle\delta\mathfrak{u}_{st},T_\epsilon\Phi\rangle &= \iint_{\Omega_\epsilon}\delta u_{st}(x)\big(\frac{u_s(y)+u_t(y)}{2}\big)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y\\
&\quad+\iint_{\Omega_\epsilon}\delta u_{st}(y)\big(\frac{u_s(x)+u_t(x)}{2}\big)T_\epsilon\Phi\,\mathrm{d}x\,\mathrm{d}y\\
&=\iint_{B_1\times\mathbb{R}^d}\boldsymbol{\tau}_{-\epsilon x_-}\delta u_{st}(x_+)\boldsymbol{\tau}_{\epsilon x_-}\big(\frac{u_s+u_t}{2}\big)(x_+)\phi(x_+)\psi(x_-)\,\mathrm{d}x_+\,\mathrm{d}x_-\\
&\quad+\iint_{B_1\times\mathbb{R}^d}\boldsymbol{\tau}_{\epsilon x_-}\delta u_{st}(x_+)\boldsymbol{\tau}_{-\epsilon x_-}\big(\frac{u_s+u_t}{2}\big)(x_+)\phi(x_+)\psi(x_-)\,\mathrm{d}x_+\,\mathrm{d}x_-\,.
\end{aligned}$$

Using again the strong continuity of $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}^d}$ in L^2, it holds

$$\begin{aligned}
\langle\delta\mathfrak{u}_{st},T_\epsilon\Phi\rangle &\to\iint_{B_1\times\mathbb{R}^d}\psi(x_-)\delta u_{st}(x_+)\big(\frac{u_s+u_t}{2}\big)(x_+)\phi(x_+)\,\mathrm{d}x_+\,\mathrm{d}x_-\\
&\qquad+\iint_{B_1\times\mathbb{R}^d}\psi(x_-)\delta u_{st}(x_+)\big(\frac{u_s+u_t}{2}\big)(x_+)\phi(x_+)\,\mathrm{d}x_+\,\mathrm{d}x_-\\
&\equiv\langle\delta(u^2)_{st},\phi\rangle\,.
\end{aligned}\tag{6.41}$$

Step 3: convergence the driver. Let $1>\delta>0$. Since $C^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we can write $u=v+w$ where $v\in C^\infty$ is such that $|v|_{L^2}\le 2|u|_{L^2}$ and $|w|_{L^2}\le\delta$. Hence for every $\delta>0$, we have

$$\begin{aligned}
&\mathfrak{u}=\mathfrak{v}+\mathfrak{w}\,,\quad\text{where }\ \mathfrak{v}\equiv v\otimes v\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)\\
&\qquad\text{and}\quad|\mathfrak{w}|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\equiv|v\otimes w+w\otimes v+w\otimes w|_{L^1}\le 4|u|_{L^2}\delta+\delta^2\le C\delta\,.
\end{aligned}\tag{6.42}$$

where we use (6.37). Since $\epsilon^{-d}\psi(\frac{x_-}{\epsilon})$ approximates the identity, changing variables as before and then using dominated convergence, we have

$$\langle S\mathfrak{v},T_\epsilon\Phi\rangle\equiv\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big(Bv(x)v(y)+v(x)Bv(y)\big)\phi(\tfrac{x+y}{2})(2\epsilon)^{-d}\psi(\tfrac{x-y}{2\epsilon})\,\mathrm{d}x\,\mathrm{d}y\to\langle\hat{B}(v^2),\phi\rangle\,,$$

and also

$$\begin{aligned}
\langle\mathbb{S}\mathfrak{v},T_\epsilon\Phi\rangle\equiv\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big(\mathbb{B}v(x)v(y)+Bv(x)Bv(y)+v(x)\mathbb{B}v(y)\big)\phi(\tfrac{x+y}{2})(2\epsilon)^{-d}\psi(\tfrac{x-y}{2\epsilon})\,\mathrm{d}x\,\mathrm{d}y\\
\to\langle\hat{\mathbb{B}}(v^2),\phi\rangle\,.
\end{aligned}$$

Using Proposition 6.1, we have

$$\begin{aligned}
\limsup_{\epsilon\to0}\langle S\mathfrak{w},T_\epsilon\Phi\rangle&\equiv\limsup_{\epsilon\to0}\langle T_\epsilon^*\mathfrak{w},T_\epsilon^{-1}S^*T_\epsilon\Phi\rangle\\
&\le C(|v|_{L^2}|w|_{L^2}+|w|_{L^2}^2)|\phi|_{W^{1,\infty}}\delta\le C'|\phi|_{W^{1,\infty}}\delta\,.
\end{aligned}$$

Similarly:

$$\limsup_{\epsilon\to0}\langle\mathbb{S}\mathfrak{w},T_\epsilon\Phi\rangle\equiv\limsup_{\epsilon\to0}\langle T_\epsilon^*\mathfrak{w},T_\epsilon^{-1}\mathbb{S}^*T_\epsilon\Phi\rangle\le C|\phi|_{W^{2,\infty}}\delta\,.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{\epsilon\to 0}\langle S\mathfrak{u}, T_\epsilon\Phi\rangle = \langle \hat{B}(u^2), \phi\rangle\,. \tag{6.43}$$

and

$$\lim_{\epsilon\to 0}\langle \mathbb{S}\mathfrak{u}, T_\epsilon\Phi\rangle = \langle \hat{\mathbb{B}}(u^2), \phi\rangle\,. \tag{6.44}$$

Conclusion. By (6.36)-(6.38) we have the following estimate, for $(s,t)\in\Delta$:

$$\langle \mathfrak{u}^{\natural,\epsilon}_{st}, \Phi\rangle \le C\left(\|u\|^2_{C(L^2)}\omega_Z(s,t)^{3\alpha} + \omega_\mu(s,t)\omega_Z(s,t)^\alpha\right)|\phi|_{W^{3,\infty}}\,.$$

From the Banach-Alaoglu theorem, and since the other terms in the equation converge, we see that for each $(s,t)\in\Delta$, there exists a linear functional $u^{2,\natural}_{st}\in (W^{3,\infty})^*$, such that

$$\langle \mathfrak{u}^{\natural,\epsilon}_{st}, \Phi\rangle \to \langle u^{2,\natural}_{st}, \phi\rangle\,.$$

for every ϕ in $W^{3,\infty}$. From (6.36) we see that $u^{2,\natural}$ belongs to $V^{1-}_{2,\mathrm{loc}}(I;(W^{3,\infty})^*)$, proving therefore that (6.40) is fulfilled. ■

We can now establish uniqueness of weak solutions in $\mathscr{B}$.

Proof of Theorem 1, uniqueness part. Testing (6.40) against $\phi := 1 \in W^{3,\infty}$, and proceeding as in Section 4, we see from the Rough Gronwall Lemma that every weak solution to (2.22) in the sense of Definition 2.2 satisfies

$$\begin{aligned}\|u\|^2_{C([0,T];L^2)} + \min(1,m)\int_0^T |\nabla u_r|^2_{L^2}\,\mathrm{d}r\\ \le C\Big(\omega_Z, |\sigma|_{W^{3,\infty}}, |\nu|_{W^{2,\infty}}, M, \|b\|_{2r,2q}, \|c\|_{r,q}, \alpha, T\Big)|u_0|^2_{L^2}\,.\end{aligned}$$

which gives (2.26). By linearity we deduce that there cannot be more than one weak solution for (2.22), hence uniqueness is proven. ■

7. Existence and stability

Finally, we intend to prove existence and stability of weak solutions to (2.22). To this end, we approximate the driving signal by smooth paths such that the classical PDE theory applies and yields existence of a unique approximate solution. The results of Section 4 yield uniform a priori estimates and the passage to the limit then follows from a compactness argument.

Let $z^n : I \to \mathbb{R}^K, n\in\mathbb{N}_0$, be a sequence of smooth paths. We define their canonical lift by $Z^n = \delta z^n$ and $\mathbb{Z}^n_{st} := \int_s^t \delta z^n_{sr}\,\mathrm{d}z^n_r$ and assume that $\mathbf{Z}^n \equiv (Z^n, \mathbb{Z}^n)$ approximates the given rough path $\mathbf{Z}\equiv(Z,\mathbb{Z})$ in the sense that

$$|Z^n - Z|_{1/\alpha-\mathrm{var}} + |\mathbb{Z}^n - \mathbb{Z}|_{2/\alpha-\mathrm{var}} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{7.1}$$

Let

$$\begin{aligned}&u^n_0 \to u_0 \ \text{ in } \ L^2\,,\\ &a^n \to a \ \text{ in } \ L^\infty\,, \qquad \text{with} \quad a^n\in\mathcal{P}_{m,M}\,,\\ &b^n \to b \ \text{ in } \ L^{2r}(I;L^{2q})\,, \quad c^n\to c \ \text{ in } \ L^r(I;L^q)\,,\\ &\sigma^n\to\sigma \ \text{ in } \ W^{3,\infty}, \qquad \nu^n\to\nu \ \text{ in } \ W^{2,\infty},\end{aligned} \tag{7.2}$$

and let

$$A^n := \partial_j\big(a^{n;ij}\partial_j\,\cdot\,\big) + b^{n;i}\partial_i + c^n\,,$$

$$B^n := Z^{n;k}(\sigma^{n;ki}\partial_i + \nu^{n;k})\,,\quad \mathbb{B}^n := \mathbb{Z}^{n;k\ell}(\sigma^{n;ki}\partial_i + \nu^{n;k})(\sigma^{n;\ell j}\partial_j + \nu^{n;\ell})\,.$$

We can assume without loss of generality that uniformly in n :

$$\begin{aligned}|u_0^n|_{L^2} + \|a^n\|_{\infty,\infty} + \|b^n\|_{2r,2q} + \|c^n\|_{r,q} + |\sigma^n|_{W^{3,\infty}} + |\nu^n|_{W^{2,\infty}} \\ \leq 1 + |u_0|_{L^2} + \|a\|_{\infty,\infty} + \|b\|_{2r,2q} + \|c\|_{r,q} + |\sigma|_{W^{3,\infty}} + |\nu|_{W^{2,\infty}}\,,\end{aligned} \tag{7.3}$$

and that

$$\left[\begin{aligned} &|B^n_{st}|_{L(W^{-k,2},W^{-k-1,2})} \leq \omega_B(s,t)^\alpha\,, \quad k\in\{0,1,2\} \\ &|\mathbb{B}^n_{st}|_{L(W^{-k,2},W^{-k-2,2})} \leq \omega_B(s,t)^{2\alpha}\,, \quad k\in\{0,1\}\,, \quad \text{for } (s,t) \text{ in } \Delta\,,\end{aligned}\right. \tag{7.4}$$

where ω_B is as in (2.21).

Recall that since z^n is smooth, existence and uniqueness of a weak solution $u^n \in \mathscr{B}_{0,T}$ to

$$\partial_t u^n = A^n u^n + \dot{B}^n u^n, \qquad u^n|_{t=0} = u_0^n,$$

in the sense of distributions, follows from the classical PDE theory (see the discussion in Section 4.1 for more details). Consequently, by Proposition 4.1, together with (7.3) and (7.4), the $\mathscr{B}_{0,T}$-norm of u^n is uniformly bounded, namely,

$$\|u^n\|^2_{\mathscr{B}_{0,T}} = \sup_{0\leq t\leq T} |u^n_t|^2_{L^2} + \int_0^T |\nabla u^n_r|^2_{L^2}\,\mathrm{d}r \leq C(1+|u_0|^2_{L^2})\,. \tag{7.5}$$

Hence the Banach-Alaoglu theorem ensures (up to a subsequence) that

$$u^n \to u \quad \text{and} \quad \nabla u^n \to \nabla u \qquad \text{weakly in } L^2([0,T]\times\mathbb{R}^d), \tag{7.6}$$

and by weak lower semicontinuity of the norm we obtain

$$\|u\|^2_{\mathscr{B}_{0,T}} < \infty\,. \tag{7.7}$$

By (7.6) and the strong convergence $\|a^n - a\|_{\infty,\infty} \to 0$ it follows that:

$$\int_s^t \langle -a_r^{n;ij}\partial_j u^n, \partial_i\phi\rangle\,\mathrm{d}r \to \int_s^t \langle -a_r^{ij}\partial_j u, \partial_i\phi\rangle\,\mathrm{d}r$$

for each ϕ in $W^{1,2}$. Moreover, using (7.2) we have

$$\|(b^n-b)\phi\|_{2,2} \leq \|b^n-b\|_{2r,2q}\|\phi\|_{\frac{2r}{r-2},\frac{2q}{q-2}} \leq \beta\|b^n-b\|_{2r,2q}|\phi|_{W^{1,2}}T^{\frac{r-2}{2r}} \to 0\,,$$

and similarly

$$\|(c^n-c)\phi\|_{2,2} \leq \beta\|c^n-c\|_{r,q}|\phi|_{W^{1,2}}T^{\frac{r-2}{2r}} \to 0\,.$$

As a consequence, using the strong/weak convergence principle, we have also

$$\int_s^t \langle b^{n;i}\partial_i u^n + c^n u^n, \phi\rangle\,\mathrm{d}r \to \int_s^t \langle b^i\partial_i u + cu, \phi\rangle\,\mathrm{d}r\,.$$

The weak convergence obtained above is however not sufficient to take the pointwise limit in time, which is needed in order to pass to the limit on the left hand side of the equation as well as in the rough integral. For that purpose, we will show that the sequence (u^n) satisfies an equicontinuity property in the space $W^{-1,2}$.

Proof of uniform equicontinuity. Using Lemma 4.1, (4.13), (7.4) and (7.5), we have the estimate

$$\begin{aligned}\Big|\!\Big|\!\Big|\int_s^t A^n u^n\,\mathrm{d}r\Big|\!\Big|\!\Big|_{-1,(2)} \leq \omega_n(s,t) \equiv (t-s)^{1/2}\mathbf{u}_n(s,t)^{1/2} + \mathbf{b}_n(s,t)^{1/(2r)}\mathbf{a}_n(s,t)^{1/2}(t-s)^{\frac{r-1}{2r}} \\ + \mathbf{c}_n(s,t)^{1/r}\mathbf{u}_n(s,t)^{\frac{r-1}{2r}}(t-s)^{\frac{r-1}{2r}}\end{aligned} \tag{7.8}$$

where we adapt the notations (4.10) in an obvious way.

Moreover, from similar computations as that of Corollary 4.1 (the proof is left to the reader) we see that u^n is a weak solution of

$$\mathrm{d}u^n = A^n u^n \,\mathrm{d}t + \mathrm{d}\mathbf{B}^n u^n \,,$$

in the sense of Definition 2.2, with respect to the scale $(W^{k,2})_{k\in\mathbb{N}_0}$. Namely:

$$\langle \delta u^n_{st}, \phi\rangle = \int_s^t \langle A^n u^n, \phi\rangle \,\mathrm{d}r + \langle B^n_{st} u^n_s, \phi\rangle + \langle \mathbb{B}^n_{st} u^n_s, \phi\rangle + \langle u^{\natural,n}_{st}, \phi\rangle \tag{7.9}$$

for each ϕ in $W^{3,\infty}$, and $(s,t)\in\Delta$. Applying Proposition 3.1 (more specifically using (3.12)), we have the bound

$$\lvert\!\lvert \delta u^n_{st}\rvert\!\rvert_{-1,(2)} \leq C\left(\omega_n(s,t) + \omega_n(s,t)^\alpha + \omega_B(s,t)^\alpha\right). \tag{7.10}$$

Now, recall that $\mathbf{a}_n(s,t)\leq C(1+2M\|u\|_{\mathscr{B}_{0,T}})\leq C_1$, and, by (2.17), that $\mathbf{u}_n(s,t)\leq C\|u\|_{\mathscr{B}_{0,T}}\leq C_2$. Using moreover (7.2), the controls $\mathbf{b}_n$ and $\mathbf{c}_n$ are equicontinuous in the sense that for each $\epsilon>0$ there exist $\delta>0$ such that

$$|s-t|\leq\delta(\epsilon) \quad\Longrightarrow\quad \mathbf{b}_n(s,t)+\mathbf{c}_n(s,t)\leq \frac{\epsilon^2}{\max(C_1,C_2)^2}\,.$$

Letting $\delta'\leq\min(\delta(\epsilon),\epsilon^2)$ and substituting in (7.8) we see that

$$\omega_n(s,t)\leq\epsilon\,,\quad \text{for all}\quad n\in\mathbb{N}_0\,,\quad \text{provided}\ |t-s|\leq\delta'\,.$$

which shows uniform equicontinuity for $\omega_n, n\geq 0$. By (7.10), the same property holds for $\lvert\!\lvert\delta u^n_{st}\rvert\!\rvert_{-1,(2)}$, hence uniform equicontinuity in $W^{-1,2}$ is proved. ∎

Thanks to the compact embedding

$$L^2(\mathbb{R}^d)\hookrightarrow W^{-1,2}_{\mathrm{loc}}(\mathbb{R}^d)\,,$$

the bound (7.5) shows that $(u^n_s)_{n\in\mathbb{N}_0}$ has a compact closure for each s in I. Using equicontinuity, a well-known infinite-dimensional version of Ascoli Theorem (we refer, e.g. to [Kel75]) ensures that, up to a subsequence:

$$u^n_s\to u_s \quad\text{in}\ W^{-1,2}_{\mathrm{loc}}(\mathbb{R}^d)\ \text{uniformly for}\quad s\in I\,. \tag{7.11}$$

By (7.6), (7.11), fixing a compactly supported ϕ in $W^{1,2}(\mathbb{R}^d)$, we have for every $(s,t)\in\Delta$:

$$\langle u^n_t - u^n_s, \phi\rangle \to \langle u_t - u_s, \phi\rangle\,.$$

Furthermore, by (2.18), for each $\phi\in W^{3,2}$ with compact support, we have

$$\lvert\!\lvert\boldsymbol{\sigma}^*\phi\rvert\!\rvert_{1,(2)}\,,\quad \lvert\!\lvert\boldsymbol{\sigma}^*\boldsymbol{\sigma}^*\phi\rvert\!\rvert_{1,(2)}\,,\quad \lvert\!\lvert\nu\boldsymbol{\sigma}^*\phi\rvert\!\rvert_{1,(2)}\,,\quad \lvert\!\lvert\boldsymbol{\sigma}^*(\nu\phi)\rvert\!\rvert_{1,(2)}\,,\ <\infty\,.$$

Finally, using (7.11) as well as (7.10), we obtain the following:

$$\begin{aligned}
Z^n_{st}\cdot\langle u^n_s,(\boldsymbol{\sigma}^n)^*\phi\rangle &\to Z_{st}\cdot\langle u_s,\boldsymbol{\sigma}^*\phi\rangle\,, & Z^n_{st}\cdot\langle u^n_s,\nu\phi\rangle &\to Z_{st}\cdot\langle u_s,\nu\phi\rangle\,,\\
\mathbb{Z}^n_{st}\cdot\langle u^n_s,(\boldsymbol{\sigma}^n)^*(\boldsymbol{\sigma}^n)^*\phi\rangle &\to \mathbb{Z}_{st}\cdot\langle u_s,\boldsymbol{\sigma}^*\boldsymbol{\sigma}^*\phi\rangle\,, & \mathbb{Z}^n_{st}\cdot\langle u^n_s,(\boldsymbol{\sigma}^n)^*(\nu\phi)\rangle &\to \mathbb{Z}_{st}\cdot\langle u_s,\boldsymbol{\sigma}^*(\nu\phi)\rangle\,,\\
\mathbb{Z}^n_{st}\cdot\langle u^n_s,\nu(\boldsymbol{\sigma}^n)^*\phi\rangle &\to \mathbb{Z}_{st}\cdot\langle u_s,\nu\boldsymbol{\sigma}^*\phi\rangle\,, & \mathbb{Z}^n_{st}\cdot\langle u^n_s,\nu^2\phi\rangle &\to \mathbb{Z}_{st}\cdot\langle u_s,\nu^2\phi\rangle\,.
\end{aligned}$$

Using in addition the estimate (3.8), we can take the limit in (7.9), so that u satisfies the corresponding weak formulation of (1.1) for every compactly supported test function in $W^{3,2}$. Due to the energy bound (7.7) we may then relax the assumptions on the test function ϕ and deduce that u is indeed a weak solution of (1.1), with respect to the scale $(W^{k,2})_{k\in\mathbb{N}_0}$.

Therefore the existence part of Theorem 1 follows. It was already shown in Section 6 that the weak solution $u \equiv \mathfrak{S}(u_0, a, b, c, \sigma, \nu, \mathbf{Z})$ is unique. In addition, due to our construction, every subsequence of $(u^n)_{n\in\mathbb{N}_0}$ contains a further subsequence which converges towards the same limit $\mathfrak{S}(u_0, a, b, c, \sigma, \nu, \mathbf{Z})$. Hence we deduce that the original sequence $(u^n)_{n\in\mathbb{N}_0}$ converges. Moreover, thanks to (7.6), (7.11), continuity of $\mathfrak{S}$ holds with respect to each of its variables. Indeed, it is enough to observe that the above proof remains applicable if $\mathbf{Z}^n$ is not necessarily a smooth approximation of $\mathbf{Z}$ in $\mathscr{C}_g$. This completes the proof of the Theorem 1 and Theorem 2. ■

APPENDIX A. AUXILIARY RESULTS

A.1. **Convergence of finite-difference approximations.** Recall (6.12). We have the following.

Lemma A.1. *Let $1 \leq p < \infty$. and fix a in $\mathbb{R}^d$. We have for every $\varphi \in W^{1,\infty}(\mathbb{R}^d)$:*

$$|\boldsymbol{\Delta}^a_\epsilon \varphi|_{L^\infty} \leq |a||\nabla\varphi|_{L^\infty} .$$

Moreover, as ϵ goes to 0, we have

$$\boldsymbol{\Delta}^a_\epsilon \varphi \to a \cdot \nabla\varphi \quad \textit{strongly in} \quad L^p(\mathbb{R}^d) ,$$

provided

- *either $p < \infty$ and $\varphi \in W^{1,p}$;*
- *or $p = \infty$ and $\varphi \in W^{2,\infty}$.*

Proof. The first bound is an easy consequence of Taylor Formula, since for every $a \in \mathbb{R}^d$

$$\boldsymbol{\Delta}^a_\epsilon \varphi(x) = a \cdot \int_0^1 \nabla\varphi\big(x + \epsilon(2\theta - 1)a\big)\,\mathrm{d}\theta . \tag{A.1}$$

Case $p \in [1, \infty)$. By Taylor Formula, we have for a.e. x in $\mathbb{R}^d$:

$$\boldsymbol{\Delta}^a_\epsilon \varphi - a \cdot \nabla\varphi(x) = a \cdot \int_0^1 \big(\nabla\varphi\big(x + \epsilon(2\theta - 1)a\big) - \nabla\varphi(x)\big)\,\mathrm{d}\theta$$

whence

$$\begin{aligned}
\int_{\mathbb{R}^d} |\boldsymbol{\Delta}^a_\epsilon \varphi - a \cdot \nabla\varphi(x)|^p\,\mathrm{d}x &\leq |a|^p \int_{\mathbb{R}^d}\int_0^1 \big|\nabla\varphi\big(x + \epsilon(2\theta - 1)a\big) - \nabla\varphi(x)\big|^p\,\mathrm{d}\theta\,\mathrm{d}x \\
&= |a|^p \int_0^1 \left(\int_{\mathbb{R}^d} \big|\nabla\varphi\big(x + \epsilon(2\theta - 1)a\big) - \nabla\varphi(x)\big|^p\,\mathrm{d}x\right)\mathrm{d}\theta \\
&= |a|^p \int_0^1 |(\boldsymbol{\tau}_{-\epsilon(2\theta-1)a} - \mathrm{id})\nabla\varphi(x)|^p_{L^p}\,\mathrm{d}\theta \\
&\to 0 , \qquad \text{as } \epsilon \to 0 ,
\end{aligned}$$

using the strong L^p continuity of $(\boldsymbol{\tau}_a)_{a\in\mathbb{R}^d}$ when $p \in [1, \infty)$ and dominated convergence.

Case $p = \infty$. Similarly, we have

$$\begin{aligned}
|\boldsymbol{\Delta}^a_\epsilon \varphi - a \cdot \nabla\varphi|_{L^\infty} &\leq \int_0^1 \sup_{x\in\mathbb{R}^d} |(\boldsymbol{\tau}_{-\epsilon(2\theta-1)a} - \mathrm{id})\nabla\varphi(x)|\,\mathrm{d}\theta \\
&\leq \int_0^1 \epsilon|2\theta - 1| \sup_{x\in\mathbb{R}^d} \int_0^1 |\nabla^2\varphi\big(x + \theta'\epsilon(2\theta - 1)a\big)|\,\mathrm{d}\theta'\,\mathrm{d}\theta \\
&\leq C\epsilon|\varphi|_{W^{2,\infty}} \to 0 , \qquad \text{as } \epsilon \to 0 ,
\end{aligned}$$

which proves the lemma. ■

A.2. **The Sewing Lemma.** A proof of the following classical result, for the case where E is a (finite-dimensional) normed vector space, can be found e.g. in [Gub04, GT10], the Banach space case being treated e.g. in [FH14]. The result appears to have an immediate extension to the case of a complete locally convex topological vector space E (l.c.v.s.), which is a repeatedly encountered scenario in PDE theory (see Remark A.1 below).

As before, we set $I := [0,T]$, for some $T > 0$, and $\Delta \equiv \Delta_I$, $\Delta^2 \equiv \Delta_I^2$ to be the corresponding simplexes. Given a l.c.v.s. E equipped with a family of seminorms $(p_\gamma)_{\gamma\in\Gamma}$, and $a > 0$ we define the space $V_1^{1/a}(I;E)$ as the set of paths $h : I \to E$ such that for every $\gamma \in \Gamma$ and every $(s,t) \in \Delta$, there holds $p_\gamma\big(\delta h_{st}\big) \le \omega_{h,\gamma}(s,t)^a$ for $(s,t) \in \Delta$, for some control depending on h and γ. Note that $V_1^{1/a}(I;E)$ is also a locally convex topological vector space given by the seminorms

$$h \mapsto \sup_{\mathfrak{p}\in\mathscr{P}(I)} \left(\sum\nolimits_{(\mathfrak{p})} p_\gamma(\delta h_{t_i t_{i+1}})^{1/a}\right)^a , \qquad \gamma \in \Gamma ,$$

(see (2.3)). The space $V_2^{1/a}(I;E)$ is defined in a similar fashion. Furthermore, $V_2^{1-}(I;E)$ corresponds to those 2-index maps $g \equiv g_{st}$ such that for each p_γ as above, there is a control $\omega_{h,\gamma}$ and $a_\gamma > 1$ with $p_\gamma(g_{st}) \le \omega_{h,\gamma}(s,t)^{a_\gamma}$ for $(s,t) \in \Delta$.

Proposition A.1 (Sewing Lemma). *Let E be a complete, locally convex topological vector space. Let $(p_\gamma)_{\gamma\in\Gamma}$ be a family of semi-norms.*

Define $\mathcal{Z}^{1-}(I;E)$ as the set of 3*-index maps $h : \Delta^2 \to E$ such that*

- *there exists a continuous $B : \Delta \to E$ with $h = \delta B$;*
- *for each $\gamma \in \Gamma$, there is a control $\omega_{h,\gamma} : \Delta \to [0,\infty)$ and $a_\gamma > 1$, such that*

$$p_\gamma\left(h_{s\theta t}\right) \le \omega_{h,\gamma}(s,t)^{a_\gamma} , \tag{A.2}$$

uniformly as $(s,\theta,t) \in \Delta^2$.

Then, there exists a linear map $\Lambda : \mathcal{Z}^{1-}(I;E) \to V_2^{1-}(I;E)$, continuous *in the sense that for every $\gamma \in \Gamma$ and $h \in \mathcal{Z}^{1-}(I;E)$ there holds*

$$p_\gamma\left(\Lambda h_{st}\right) \le C_{a_\gamma}\omega_{h,\gamma}(s,t)^{a_\gamma} , \quad \textit{for every} \quad (s,t) \in \Delta , \tag{A.3}$$

where the above constant only depends on the value of $a_\gamma > 1$. In addition, Λ is a right inverse for δ, namely

$$\delta\Lambda = \mathrm{id}\,|_{\mathcal{Z}^{1-}} , \tag{A.4}$$

and it is unique *in the class of linear mappings fulfilling the properties* (A.3)-(A.4).

Finally, for any $(s,t) \in \Delta$, we have the explicit formula:

$$\Lambda_{st} h = \lim_{|\mathfrak{p}|\to 0} \left(B_{st} - \sum\nolimits_{(\mathfrak{p})} B_{t_i t_{i+1}}\right) , \tag{A.5}$$

where we use the summation convention (2.3).

Example A.1. The above infinite-dimensional Sewing Lemma applies in $\mathscr{D}'(\mathcal{O})$, the space of distributions over an open subset $\mathcal{O}$ of some Euclidean space, for which a family of semi-norms is provided by

$$p_\phi(v) := |\langle v, \phi\rangle| , \qquad \phi \in C_c^\infty(\mathcal{O}) ,$$

for v in $\mathscr{D}'(\mathcal{O})$.

We could replace $\mathscr{D}'$ by the space of Schwarz distributions $\mathscr{S}'$, or any Banach space of linear functionals endowed with the weak-* topology.

Proof. The proof is similar to that of [FH14]. Fix $(s,t) \in \Delta$, and consider a partition $\mathfrak{p} := \{s \equiv t_1 < t_2 < \cdots < t_k \equiv t\}$ of $[s,t]$, such that $\#\mathfrak{p} = k \geq 2$. Define

$$\Lambda^{\mathfrak{p}} h := B_{st} - \sum_{1 \leq i \leq k-1} B_{t_i t_{i+1}} \,,$$

where B is such that $\delta B = h$.

Let $\gamma \in \Gamma$. By the superadditivity of $\omega_{h,\gamma}$, there exists $i_1 \in \{1, \ldots, k\}$ such that

$$\omega_{h,\gamma}(t_{i_1-1}, t_{i_1+1}) \leq \frac{2}{k-1} \omega_{h,\gamma}(s,t) \,.$$

Moreover, we have the relation

$$p_\gamma\big(\Lambda^{\mathfrak{p}\setminus\{t_{i_1}\}} h - \Lambda^{\mathfrak{p}} h\big) = p_\gamma(\delta B_{t_{i-1}, t_i, t_{i+1}}) \leq \left(\frac{2}{k-1}\omega_{h,\gamma}(s,t)\right)^{a_\gamma} \,. \tag{A.6}$$

Replacing $\mathfrak{p}$ by $\mathfrak{p} \setminus \{t_{i_1}\}$, we can iterate this procedure until we end up with the trivial partition $\mathfrak{p} \setminus \{t_{i_1}, \ldots, t_{i_{k-2}}\} \equiv \{s,t\}$ for which $\Lambda^{\{s,t\}} h = 0$ (note that the order in which the points t_i are dropped out may depend on γ in Γ, but this is not a problem since the final expression does not). Writing that

$$\Lambda^{\mathfrak{p}} h = \big(\Lambda^{\mathfrak{p}} - \Lambda^{\mathfrak{p}\setminus\{t_{i_1}\}}\big)\, h + \cdots + \left(\Lambda^{\mathfrak{p}\setminus\{t_{i_1}, \ldots, t_{i_{k-3}}\}} - \Lambda^{\{s,t\}}\right) h \,,$$

and using (A.6) $k-2$ times, we find the maximal inequality

$$p_\gamma\left(\Lambda^{\mathfrak{p}} h\right) \leq 2^{a_\gamma} \omega_{h,\gamma}(s,t)^{a_\gamma} \sum_{i=1}^{k-2} i^{-a_\gamma} \leq 2^{a_\gamma} \omega_{h,\gamma}(s,t)^{a_\gamma} \sum_{i=1}^{\infty} i^{-a_\gamma} \leq C_{a_\gamma} \omega_{h,\gamma}(s,t)^{a_\gamma} \,, \tag{A.7}$$

and this holds for every γ in Γ.

Now, let us consider a refined partition $\mathfrak{p}' \subset \mathfrak{p}$. We have

$$\Lambda^{\mathfrak{p}} h - \Lambda^{\mathfrak{p}'} h = - \sum_{1 \leq i \leq k-1} \underbrace{\left(B_{t_i t_{i+1}} - \sum_{\{\theta,\tau\} \subset \mathfrak{p}' \cap [t_i, t_{i+1}],\, \theta < \tau} B_{\theta\tau} \right)}_{\Lambda^{\mathfrak{p}' \cap [t_i, t_{i+1}]} h}$$

whence, using the maximal inequality (A.7) on each $[t_i, t_{i+1}]$, there comes:

$$p_\gamma\Big(\Lambda^{\mathfrak{p}} h - \Lambda^{\mathfrak{p}'} h\Big) \leq \sum_{t_i \in \mathfrak{p},\, i<k} C_a \omega_{h,\gamma}(t_i, t_{i+1})^{a_\gamma} \,.$$

Since $a_\gamma > 1$, the r.h.s. above vanishes as the size of $\mathfrak{p}$ goes to 0, which by completeness of E shows the convergence of $\Lambda^{\mathfrak{p}} h$ towards some $\Lambda_{st} h$, for any $(s,t) \in \Delta$.

Finally, one can follow the lines of *Step 4* in [GT10, Proposition 2.3] to show that we have $\delta \Lambda h = h$. This completes the proof. ■

Corollary and definition A.1. *Given $\alpha \in (0,1]$, let B in $V_2^{1/\alpha}(I;E)$ and assume that $\delta B \in \mathcal{Z}^{1-}$. Define*

$$\mathcal{I}(B) := B - \Lambda \delta B \, \in V_2^{1/\alpha}(I;E) \,. \tag{A.8}$$

Then, the linear map $\mathcal{I} : V_2^{1/\alpha}(I;E) \to V_2^{1/\alpha}(I;E)$, $B \mapsto \mathcal{I}(B)$ fulfills the following properties

- $\delta \mathcal{I} = 0$;
- *if $h \in V_2^{1/\alpha}(I;E)$ is another 2-index map such that $\delta h = 0$ and $h - B \in V^{1-}(I;E)$, then $h = \mathcal{I}(B)$;*

- *for any B as above, $\mathcal{I}(B)$ is given by*

$$\mathcal{I}_{st}B = \lim_{|\mathfrak{p}|\to 0} \sum\nolimits_{(\mathfrak{p})} B_{t_i t_{i+1}}\,; \tag{A.9}$$

- *let E be a reflexive Banach space, and assume that $f : I \to L(E, F)$, $g : I \to E$ are measurable maps, f being continuous, and such that g belongs to $\mathcal{AC}(I; E)$. Let $\dot{g} \in L^1(I; E)$ denote the weak derivative of the path g. Assume in addition that $\delta(f\delta g) \in \mathcal{Z}^{1-}(I; F)$. Then, we have $\int_I |f_r \dot{g}_r|_F \,\mathrm{d}r < \infty$ and*

$$\mathcal{I}(f\delta g)_{st} = \int_s^t f_r \dot{g}_r \,\mathrm{d}r \quad \textit{(Bochner integral in } F\textit{)}\,, \tag{A.10}$$

where $f\delta g$ is to be understood as the map $(s,t) \in \Delta \mapsto f_s \delta g_{st}$.

For B as above, the 2-index map $(s,t) \in \Delta \mapsto \mathcal{I}(B)_{st}$ is called the rough integral *of B.*

Proof. The three first statements are immediate consequences of Proposition A.1, (for a proof in the Banach space setting, we refer e.g. to [Gub04, FH14]).

Let us check the last point. First, note that the weak derivative of g exists, because any reflexive space fulfills the Radon-Nikodym property (see [DU77, Definition 3 p. 61 and Corollary 13 p. 76]). From the formula (A.9), it holds that $\mathcal{I}(f\delta g)$ is the limit, as $n \to \infty$ of the partial sums

$$I_n := \sum_{(\mathfrak{p}_n)} f_{t_i^n} \delta g_{t_i^n t_{i+1}^n} \equiv \sum_{(\mathfrak{p}_n)} f_{t_i^n} \int_{t_i^n}^{t_{i+1}^n} \dot{g}_r \,\mathrm{d}r = \int_I f_r \dot{g}_r \,\mathrm{d}r - \sum_{(\mathfrak{p}_n)} \int_{t_i^n}^{t_{i+1}^n} (f_r - f_{t_i^n}) \dot{g}_r \,\mathrm{d}r\,,$$

where $\mathfrak{p}_n \equiv (t_i^n)$ is such that $|\mathfrak{p}_n| \to 0$. The mapping $f : I \equiv [0,T] \to \mathbb{R}$ is continuous, hence uniformly continuous, so that the second term above goes to 0 as $n \to \infty$. Therefore, $\mathcal{I}(f\delta g) \equiv \lim I_n = \int_I f_r \dot{g}_r \,\mathrm{d}r$, which proves (A.10). ∎

A.3. **Families of smoothing operators.** Let R_η denote the family of smoothing operators defined on $\varphi \in W^{k,\infty} \equiv W^{k,\infty}(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, by

$$R_\eta \varphi(x) := [\varphi * \varrho_\eta](x) = \left[\varphi * \varrho\Big(\frac{\cdot}{\eta}\Big)\eta^{-d}\right](x) \equiv \int_{\mathbb{R}^d} \varphi(\xi) \varrho\Big(\frac{x-\xi}{\eta}\Big)\frac{\mathrm{d}\xi}{\eta^d}, \quad x \in \mathbb{R}^d\,, \tag{A.11}$$

where $\varrho \in C^\infty(\mathbb{R}^d; \mathbb{R})$ is a non-negative, *radially symmetric* function that integrates to 1, and such that $\operatorname{Supp} \varrho \subset B_1$. As a consequence, R_η reproduces affine linear functions exactly and it is then possible to recover the error of order η^2 for $|(R_\eta - \mathrm{id})\varphi|_{L^\infty}$ provided φ belongs to $W^{2,\infty}$ (this is classical and follows from a Taylor expansion of the integrand). More precisely, we have the following.

Lemma A.2. *The family $(R_\eta)_{\eta\in(0,1)}$ is a 2-step family of smoothing operators over the scale $W^{k,\infty}(\mathbb{R}^d)$.*

Remark A.1. One could also consider different mollifiers (no longer nonnegative) which would reproduce polynomials of higher order exactly, in order to obtain higher rates of convergence of $|(R_\eta - \mathrm{id})\varphi|_{W^{n,\infty}}$ under suitable regularity assumption on φ. Second order estimates in η are however sufficient here.

Since R_η increases the support of test functions, it cannot define a smoothing family for the scale $(\mathcal{F}_k)_{k\in\mathbb{N}_0}$ defined in (5.4). To deal with that problem, we need to introduce a suitable cut-off function. Let $\theta_\eta \in C_c^\infty(\mathbb{R})$ such that

$$0 \le \theta_\eta \le 1\,, \quad \operatorname{Supp}\theta_\eta \subset B_{1-2\eta} \subset \mathbb{R}\,, \quad \theta \equiv 1 \quad \text{on} \quad B_{1-3\eta} \subset \mathbb{R}\,, \tag{A.12}$$

and such that for $k=1,2$:

$$|\nabla^k\theta_\eta| \leq \frac{C}{\eta^k}\,.$$

Next, we define

$$\Theta_\eta(x) := \theta_\eta(|x|^2)\,,\quad \text{for}\quad x\in\mathbb{R}^d\,. \tag{A.13}$$

The following has been shown in [DGHT16a].

Lemma A.3. *There is a constant $C_\theta > 0$ such that for $k = 0, 1, 2,$ and every ψ in $W^{k,\infty}(\mathbb{R}^d)$ compactly supported in B_1 :*

$$|\Theta_\eta\psi|_{W^{k,\infty}} \leq C_\theta|\psi|_{W^{k,\infty}}\,. \tag{A.14}$$

If in addition we assume $\psi\in W^{k,\infty}(\mathbb{R}^d)$, with $0\leq \ell\leq k\leq 3$ then

$$|(1-\Theta_\eta)\psi|_{W^{\ell,\infty}} \leq C_\theta\eta^{k-\ell}|\psi|_{W^{k,\infty}}\,. \tag{A.15}$$

Corollary A.1. *The linear mappings $J_\eta : \mathcal{F}_0(\Omega)\to\mathcal{F}_0(\Omega), \eta\in(0,1)$, defined by*

$$J_\eta\phi := \chi\circ\big(R_\eta\otimes(R_\eta\Theta_\eta)(\phi\circ\chi^{-1})\big)$$

where we keep the notations of Lemma A.2, (6.13) *and* (A.13)*, provide a 2-step family of smoothing operators with respect to the scale $(\mathcal{F}_k(\Omega))_{k\in\mathbb{N}_0}$.*

Proof. Since $\sqrt{2}\chi$ is a rotation, it is sufficient to show the corollary on the scale

$$F_k := \big\{\phi\in W^{k,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\,, \operatorname{Supp}\phi\subset\mathbb{R}^d\times B_1\big\}\ , \tag{A.16}$$

endowed with the norm $(\!|\cdot|\!)_k := |\cdot|_{W^{k,\infty}}$, and $J_\eta := R_\eta\otimes(R_\eta\Theta_\eta)$.

Note first that for any fixed $x\in\mathbb{R}^d$, and $\phi\in F_k$:

$$\operatorname{Supp}\big(\operatorname{id}\otimes(R_\eta\Theta_\eta)\phi(x,\cdot)\big) \subset \operatorname{Supp}(\Theta_\eta\phi(x,\cdot)) + \operatorname{Supp}(\varrho_\eta) \subset B_1 \tag{A.17}$$

Since we have $J_\eta\phi = (R_\eta\otimes\operatorname{id})(\operatorname{id}\otimes R_\eta\Theta_\eta)\phi$, we see that

$$\operatorname{Supp} J_\eta\phi\subset B_1\,,$$

and because $J_\eta\phi$ is smooth, the property (J1) follows.

Concerning (J2), let for instance fix $k=0$, and $\phi\in F_0$. Using Lemma A.2, denoting by $\psi^y := (\operatorname{id}\otimes R_\eta\Theta_\eta)\phi(\cdot,y)$, we have for any $1\leq i\leq d$ and $x,y\in\mathbb{R}^d$:

$$\begin{aligned}|\partial_{x_i}J_\eta\phi(x,y)| \equiv |\partial_{x_i}(R_\eta\otimes\operatorname{id})[\psi^y](x)| \leq \frac{C}{\eta}|\psi^y|_{L^\infty_x} \leq \frac{C}{\eta}\int_{\mathbb{R}^d}\Theta_\eta(y')|\phi(\cdot,y')|_{L^\infty_x}\varrho_\eta(y-y')\,\mathrm{d}y'\\ \leq \frac{C}{\eta}(\!|\phi|\!)_0\,.\end{aligned}$$

Similarly, denoting by $\tilde{\psi}^x := (R_\eta\otimes\operatorname{id})(1-\Theta_\eta)\phi(x,\cdot)$, it holds

$$\begin{aligned}|\partial_{y_i}J_\eta\phi(x,y)| &\leq |\partial_{y_i}(R_\eta\otimes R_\eta)\phi| + |\partial_{y_i}(\operatorname{id}\otimes R_\eta)\tilde{\psi}^x(y)| \leq \frac{C}{\eta}(\!|\phi|\!)_0 + \frac{C}{\eta}|\tilde{\psi}^x|_{L^\infty_y}\\ &\leq \frac{C}{\eta}(\!|\phi|\!)_0 + \frac{C}{\eta}\int_{\mathbb{R}^d}|(1-\Theta_\eta(y))\phi(x',\cdot)|_{L^\infty_y}\varrho_\eta(x-x')\,\mathrm{d}x' \leq \frac{C'}{\eta}(\!|\phi|\!)_0\,.\end{aligned}$$

Inequalities corresponding to $k=1,2$ are shown in a similar way, using in addition (A.14)-(A.15). The bounds related to (J3) are similar. ∎

REFERENCES

[BG15] Ismaël Bailleul and Massimiliano Gubinelli. Unbounded rough drivers. *arXiv preprint arXiv:1501.02074*, 2015.

[Bre10] Haïm Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[Cat15] Rémi Catellier. Rough linear transport equation with an irregular drift. *arXiv preprint arXiv:1501.03000*, 2015.

[CF09] Michael Caruana and Peter Friz. Partial differential equations driven by rough paths. *Journal of Differential Equations*, **247**(1):140–173, 2009.

[CFO11] Michael Caruana, Peter K Friz, and Harald Oberhauser. A (rough) pathwise approach to a class of non-linear stochastic partial differential equations. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, volume **28**, pages 27–46. Elsevier, 2011.

[Dav07] A.M. Davie. Differential equations driven by rough paths: an approach via discrete approximation. In *Appl. Math. Res. Express*. Citeseer, 2007.

[DFS14] Joscha Diehl, Peter K. Friz, and Wilhelm Stannat. Stochastic partial differential equations: a rough path view. *arXiv preprint arXiv:1412.6557*, 2014.

[DGHT16a] Aurélien Deya, Massimiliano Gubinelli, Martina Hofmanova, and Samy Tindel. A priori estimates for rough PDEs with application to rough conservation laws. *arXiv preprint arXiv:1604.00437*, 2016.

[DGHT16b] Aurélien Deya, Massimiliano Gubinelli, Martina Hofmanova, and Samy Tindel. One-dimensional reflected rough differential equations. *arXiv preprint arXiv:1610.07481*, 2016.

[DGT12] Aurélien Deya, Massimiliano Gubinelli, and Samy Tindel. Non-linear rough heat equations. *Probability Theory and Related Fields*, **153**(1-2):97–147, 2012.

[DU77] J. Diestel and J.J. Uhl. Vector Measures. *American Mathematical Society, Rhode Island*, 1977.

[FH14] Peter K. Friz and Martin Hairer. *A course on rough paths: with an introduction to regularity structures*. Springer, 2014.

[FV10] Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths: theory and applications*, volume 120. Cambridge University Press, 2010.

[GIP15] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. In *Forum of Mathematics, Pi*, volume **3**, page e6. Cambridge Univ Press, 2015.

[GT10] Massimiliano Gubinelli and Samy Tindel. Rough evolution equations. *The Annals of Probability*, **38**(1):1–75, 2010.

[GTT14] Massimiliano Gubinelli, Samy Tindel, and I. Torrecilla. Controlled viscosity solutions of fully nonlinear rough PDEs. *arXiv preprint arXiv:1403.2832*, 2014.

[Gub04] Massimiliano Gubinelli. Controlling rough paths. *Journal of Functional Analysis*, 216:86–140, 2004.

[Hai14] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, **198**(2):269–504, 2014.

[Kel75] John L. Kelley. *General topology*, volume **27**. Springer Science & Business Media, 1975.

[LSU68] O. Ladyzhenskaya, V. Solonnikov, and N. Uraltseva. Linear and quasilinear parabolic equations of second order. *Translation of Mathematical Monographs, AMS, Rhode Island*, 1968.

[Lun09] Alessandra Lunardi. *Interpolation theory*. Edizioni della normale, 2009.

[Lyo98] Terry J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, **14**(2):215–310, 1998.

[OW16] F. Otto and H. Weber. Quasilinear SPDEs via rough paths. *ArXiv preprint arXiv:1605.09744*, 2016.

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